Chapter 1
WATER WAVE MECHANICS

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Water Wave Mechanics

II-1-1. Introduction

a. Waves on the surface of the ocean with periods of 3 to 25 sec are primarily generated by winds and are a fundamental feature of coastal regions of the world. Other wave motions exist on the ocean including internal waves, tides, and edge waves. For the remainder of this chapter, unless otherwise indicated, the term waves will apply only to surface gravity waves in the wind wave range of 3 to 25 sec.

b. Knowledge of these waves and the forces they generate is essential for the design of coastal projects since they are the major factor that determines the geometry of beaches, the planning and design of marinas, waterways, shore protection measures, hydraulic structures, and other civil and military coastal works. Estimates of wave conditions are needed in almost all coastal engineering studies. The purpose of this chapter is to give engineers theories and mathematical formulae for describing ocean surface waves and the forces, accelerations, and velocities due to them. This chapter is organized into two sections: *Regular Waves* and *Irregular Waves*.

c. In the *Regular Waves* section, the objective is to provide a detailed understanding of the mechanics of a wave field through examination of waves of constant height and period. In the *Irregular Waves* section, the objective is to describe statistical methods for analyzing irregular waves (wave systems where successive waves may have differing periods and heights) which are more descriptive of the waves seen in nature.

d. In looking at the sea surface, it is typically irregular and three-dimensional (3-D). The sea surface changes in time, and thus, it is unsteady. At this time, this complex, time-varying 3-D surface cannot be adequately described in its full complexity; neither can the velocities, pressures, and accelerations of the underlying water required for engineering calculations. In order to arrive at estimates of the required parameters, a number of simplifying assumptions must be made to make the problems tractable, reliable and helpful through comparison to experiments and observations. Some of the assumptions and approximations that are made to describe the 3-D, time-dependent complex sea surface in a simpler fashion for engineering works may be unrealistic, but necessary for mathematical reasons.

e. The *Regular Waves* section of this chapter begins with the simplest mathematical representation assuming ocean waves are two-dimensional (2-D), small in amplitude, sinusoidal, and progressively definable by their wave height and period in a given water depth. In this simplest representation of ocean waves, wave motions and displacements, kinematics (that is, wave velocities and accelerations), and dynamics (that is, wave pressures and resulting forces and moments) will be determined for engineering design estimates. When wave height becomes larger, the simple treatment may not be adequate. The next part of the *Regular Waves* section considers 2-D approximation of the ocean surface to deviate from a pure sinusoid. This representation requires using more mathematically complicated theories. These theories become nonlinear and allow formulation of waves that are not of purely sinusoidal in shape; for example, waves having the flatter troughs and peaked crests typically seen in shallow coastal waters when waves are relatively high.

f. The *Irregular Waves* section of this chapter is devoted to an alternative description of ocean waves. Statistical methods for describing the natural time-dependent three-dimensional characteristics of real wave systems are presented. A complete 3-D representation of ocean waves requires considering the sea surface as an irregular wave train with random characteristics. To quantify this randomness of ocean waves, the *Irregular Waves* section employs statistical and probabilistic theories. Even with this approach, simplifications are required. One approach is to transform the sea surface using Fourier theory into summation of simple sine waves and then to define a wave's characteristics in terms of its spectrum. This allows treatment of the variability of waves with respect to period and direction of travel. The second approach is to describe a wave record at a point as a sequence of individual waves with different heights and periods and then to consider the variability of the wave field in terms of the probability of individual waves.

g. At the present time, practicing coastal engineers must use a combination of these approaches to obtain information for design. For example, information from the *Irregular Waves* section will be used to determine the expected range of wave conditions and directional distributions of wave energy in order to select an individual wave height and period for the problem under study. Then procedures from the *Regular Waves* section will be used to characterize the kinematics and dynamics that might be expected. However, it should be noted that the procedures for selecting and using irregular wave conditions remain an area of some uncertainty.

h. The major generating force for waves is the wind acting on the air-sea interface. A significant amount of wave energy is dissipated in the nearshore region and on beaches. Wave energy forms beaches; sorts bottom sediments on the shore face; transports bottom materials onshore, offshore, and alongshore; and exerts forces upon coastal structures. A basic understanding of the fundamental physical processes in the generation and propagation of surface waves must precede any attempt to understand complex water motion in seas, lakes and waterways. The *Regular Waves* section of this chapter outlines the fundamental principles governing the mechanics of wave motion essential in the planning and design of coastal works. The *Irregular Waves* section of this chapter discusses the applicable statistical and probabilistic theories.

i. Detailed descriptions of the basic equations for water mechanics are available in several textbooks (see for example, Kinsman 1965; Stoker 1957; Ippen 1966; Le Méhauté 1976; Phillips 1977; Crapper 1984; Mei 1991; Dean and Dalrymple 1991). The Regular Waves section of this chapter provides only an introduction to wave mechanics, and it focuses on simple water wave theories for coastal engineers. Methods are discussed for estimating wave surface profiles, water particle motion, wave energy, and wave transformations due to interaction with the bottom and with structures.

j. The simplest wave theory is the *first-order, small-amplitude*, or *Airy* wave theory which will hereafter be called *linear theory*. Many engineering problems can be handled with ease and reasonable accuracy by this theory. For convenience, prediction methods in coastal engineering generally have been based on simple waves. For some situations, simple theories provide acceptable estimates of wave conditions.

k. When waves become large or travel toward shore into shallow water, higher-order wave theories are often required to describe wave phenomena. These theories represent *nonlinear waves*. The linear theory that is valid when waves are infinitesimally small and their motion is small also provides some insight for finite-amplitude periodic waves (nonlinear). However, the linear theory cannot account for the fact that wave crests are higher above the mean water line than the troughs are below the mean water line. Results obtained from the various theories should be carefully interpreted for use in the design of coastal projects or for the description of coastal environment.

l. Any basic physical description of a water wave involves both its surface form and the water motion beneath the surface. A wave that can be described in simple mathematical terms is called a *simple wave*. Waves comprised of several components and difficult to describe in form or motion are termed *wave trains* or *complex waves*. Sinusoidal or monochromatic waves are examples of simple waves, since their surface profile can be described by a single sine or cosine function. A wave is *periodic* if its motion and surface profile recur in equal intervals of time termed the *wave period*. A wave form that moves horizontally relative to a fixed point is called a *progressive wave* and the direction in which it moves is termed the *direction of wave propagation*. A progressive wave is called *wave of permanent form* if it propagates without experiencing any change in shape.

m. Water waves are considered *oscillatory* or *nearly oscillatory* if the motion described by the water particles is circular orbits that are closed or nearly closed for each wave period. The linear theory represents pure oscillatory waves. Waves defined by finite-amplitude wave theories are not pure oscillatory waves but still periodic since the fluid is moved in the direction of wave advance by each successive wave. This motion is termed *mass transport* of the waves. When water particles advance with the wave and do not return to their original position, the wave is called a *wave of translation*. A solitary wave is an example of a wave of translation.

n. It is important in coastal practice to differentiate between two types of surface waves. These are *seas* and *swells*. Seas refer to short-period waves still being created by winds. Swells refer to waves that have moved out of the generating area. In general, swells are more regular waves with well-defined long crests and relatively long periods.

o. The growth of wind-generated oceanic waves is not indefinite. The point when waves stop growing is termed a *fully developed sea* condition. Wind energy is imparted to the water leading to the growth of waves; however, after a point, the energy imparted to the waters is dissipated by wave breaking. Seas are short-crested and irregular and their periods are within the 3- to 25- sec range. Seas usually have shorter periods and lengths, and their surface appears much more disturbed than for swells. Waves assume a more orderly state with the appearance of definite crests and troughs when they are no longer under the influence of winds (swell).

p. To an observer at a large distance from a storm, swells originating in a storm area will appear to be almost unidirectional (i.e., they propagate in a predominant direction) and long-crested (i.e., they have well-defined and distinctly separated crests). Although waves of different periods existed originally together in the generation area (seas), in time the various wave components in the sea separate from one another. Longer period waves move faster and reach distant sites first. Shorter period components may reach the site several days later. In the wave generation area, energy is transferred from shorter period waves to the longer waves. Waves can travel hundreds or thousands of kilometers without much loss of energy. However, some wave energy is dissipated internally within the fluid, by interaction with the air above, by turbulence upon breaking, and by percolation and friction with the seabed. Short-period components lose their energy more readily than long-period components. As a consequence of these processes, the periods of swell waves tend to be somewhat longer than seas. Swells typically have periods greater than 10 sec.

II-1-2. Regular Waves

a. Introduction. Wave theories are approximations to reality. They may describe some phenomena well under certain conditions that satisfy the assumptions made in their derivation. They may fail to describe other phenomena that violate those assumptions. In adopting a theory, care must be taken to ensure that the wave phenomenon of interest is described reasonably well by the theory adopted, since shore protection design depends on the ability to predict wave surface profiles and water motion, and on the accuracy of such predictions.

b. Definition of wave parameters.

(1) A progressive wave may be represented by the variables x (spatial) and t (temporal) or by their combination (phase), defined as $\theta = kx - \omega t$, where k and ω are described in the following paragraphs. The values of θ vary between 0 and 2π. Since the θ-representation is a simple and compact notation, it will be used in this chapter. Figure II-1-1 depicts parameters that define a simple, progressive wave as it passes a fixed point in the ocean. A simple, periodic wave of permanent form propagating over a horizontal bottom may be completely characterized by the wave height H wavelength L and water depth d.

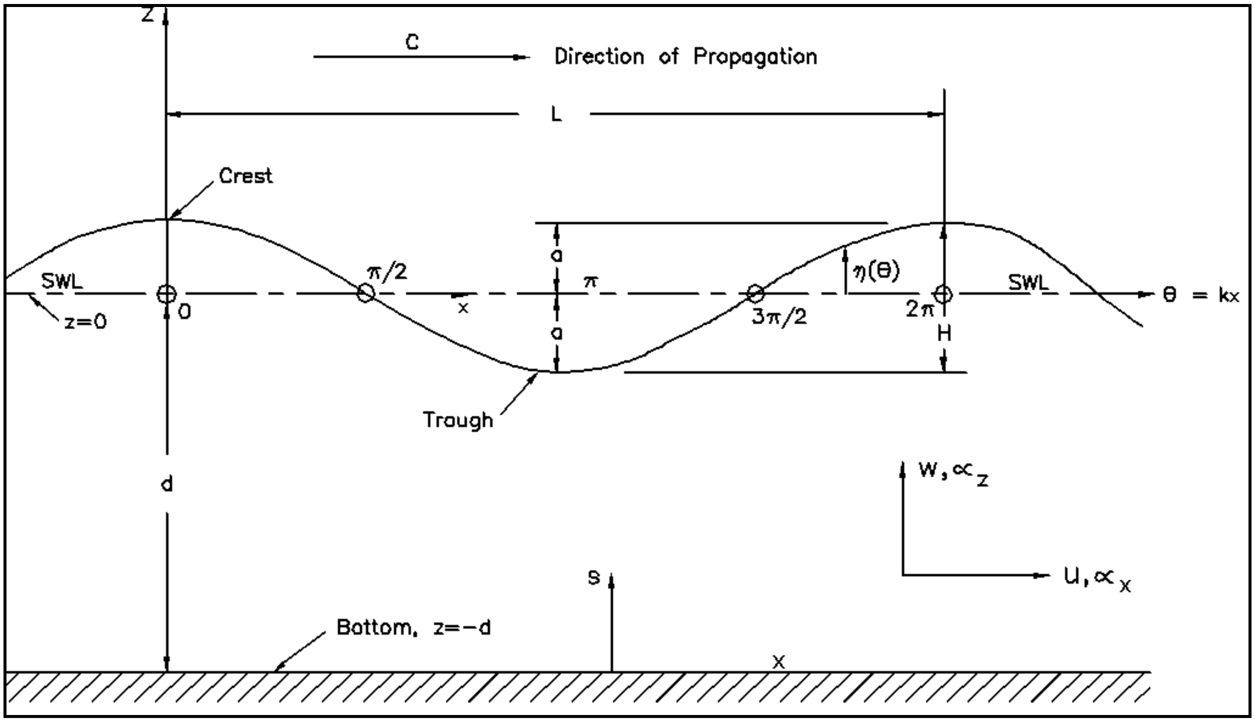

Figure II-1-1. Definition of terms - elementary, sinusoidal, progressive wave

(2) As shown in Figure II-1-1, the highest point of the wave is the *crest* and the lowest point is the *trough*. For linear or small-amplitude waves, the height of the crest above the still-water level (SWL) and the distance of the trough below the SWL are each equal to the wave amplitude a. Therefore $a = H/2$, where H = *the wave height*. The time interval between the passage of two successive wave crests or troughs at a given point is the *wave period* T. The *wavelength* L is the horizontal distance between two identical points on two successive wave crests or two successive wave troughs.

(3) Other wave parameters include $\omega = 2\pi/T$ the *angular* or *radian frequency*, the *wave number* $k = 2\pi/L$, the *phase velocity* or *wave celerity* $C = L/T = \omega/k$, the *wave steepness* $\varepsilon = H/L$, the *relative depth* d/L, and the *relative wave height* H/d. These are the most common parameters encountered in coastal practice. Wave motion can be defined in terms of dimensionless parameters H/L, H/d, and d/L; these are often used in practice. The dimensionless parameters ka and kd, preferred in research works, can be substituted for H/L and d/L, respectively, since these differ only by a constant factor 2π from those preferred by engineers.

c. Linear wave theory.

(1) Introduction.

(a) The most elementary wave theory is the *small-amplitude* or *linear wave theory*. This theory, developed by Airy (1845), is easy to apply, and gives a reasonable approximation of wave characteristics for a wide range of wave parameters. A more complete theoretical description of waves may be obtained as the sum of many successive approximations, where each additional term in the series is a correction to preceding terms. For some situations, waves are better described by these higher-order theories, which are usually referred to as *finite-amplitude wave theories* (Mei 1991, Dean and Dalrymple 1991). Although there are limitations to its applicability, linear theory can still be useful provided the assumptions made in developing this simple theory are not grossly violated.

(b) The assumptions made in developing the linear wave theory are:

- The fluid is homogeneous and incompressible; therefore, the density ρ is a constant.
- Surface tension can be neglected.
- Coriolis effect due to the earth's rotation can be neglected.
- Pressure at the free surface is uniform and constant.
- The fluid is ideal or inviscid (lacks viscosity).
- The particular wave being considered does not interact with any other water motions. The flow is irrotational so that water particles do not rotate (only normal forces are important and shearing forces are negligible).
- The bed is a horizontal, fixed, impermeable boundary, which implies that the vertical velocity at the bed is zero.
- The wave amplitude is small and the waveform is invariant in time and space.
- Waves are plane or long-crested (two-dimensional).

(c) The first three assumptions are valid for virtually all coastal engineering problems. It is necessary to relax the fourth, fifth, and sixth assumptions for some specialized problems not considered in this manual. Relaxing the three final assumptions is essential in many problems, and is considered later in this chapter.

(d) The assumption of irrotationality stated as the sixth assumption above allows the use of a mathematical function termed the *velocity potential* Φ. The velocity potential is a scaler function whose gradient (i.e., the rate of change of Φ relative to the x-and z-coordinates in two dimensions where x = horizontal, z = vertical) at any point in fluid is the velocity vector. Thus,

$$u = \frac{\partial \Phi}{\partial x} \tag{II-1-1}$$

is the fluid velocity in the x-direction, and

$$w = \frac{\partial \Phi}{\partial z} \tag{II-1-2}$$

is the fluid velocity in the z-direction. Φ has the units of length squared divided by time. Consequently, if $\Phi(x, z, t)$ is known over the flow field, then fluid particle velocity components *u* and *w* can be found.

(e) The incompressible assumption (a) above implies that there is another mathematical function termed the *stream function* Ψ. Some wave theories are formulated in terms of the stream function Ψ, which is orthogonal to the potential function Φ. Lines of constant values of the potential function (equipotential lines) and lines of constant values of the stream function are mutually perpendicular or orthogonal. Consequently, if Φ is known, Ψ can be found, or vice versa, using the equations

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial z} \tag{II-1-3}$$

$$\frac{\partial \Phi}{\partial z} = -\frac{\partial \Psi}{\partial x} \tag{II-1-4}$$

termed the *Cauchy-Riemann conditions* (Whitham 1974; Milne-Thompson 1976). Both Φ and Ψ satisfy the *Laplace equation* which governs the flow of an *ideal fluid* (inviscid and incompressible fluid). Thus, under the assumptions outlined above, the Laplace equation governs the flow beneath waves. The Laplace equation in two dimensions with x = horizontal, and z = vertical axes in terms of velocity potential Φ is given by

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{II-1-5}$$

(f) In terms of the stream function, Ψ, Laplace's equation becomes

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial z^2} = 0 \tag{II-1-6}$$

(g) The linear theory formulation is usually developed in terms of the potential function, Φ.

In applying the seventh assumption to waves in water of varying depth (encountered when waves approach a beach), the local depth is usually used. This can be justified, but not without difficulty, for most practical cases in which the bottom slope is flatter than about 1 on 10. A progressive wave moving into shallow water will change its shape significantly. Effects due to the wave transformations are addressed in Parts II-3 and II-4.

(h) The most fundamental description of a simple sinusoidal oscillatory wave is by its length L (the horizontal distance between corresponding points on two successive waves), height H (the vertical distance to its crest from the preceding trough), period T (the time for two successive crests to pass a given point), and depth d (the distance from the bed to SWL).

(i) Figure II-1-1 shows a two-dimensional, simple progressive wave propagating in the positive x-direction, using the symbols presented above. The symbol η denotes the displacement of the water surface relative to the SWL and is a function of x and time t. At the wave crest, η is equal to the amplitude of the wave a, or one-half the wave height $H/2$.

(2) Wave celerity, length, and period.

(a) The speed at which a wave form propagates is termed the *phase velocity* or *wave celerity* C. Since the distance traveled by a wave during one wave period is equal to one wavelength, wave celerity can be related to the wave period and length by

$$C = \frac{L}{T} \tag{II-1-7}$$

(b) An expression relating wave celerity to wavelength and water depth is given by

$$C = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)} \tag{II-1-8}$$

(c) Equation II-1-8 is termed the *dispersion relation* since it indicates that waves with different periods travel at different speeds. For a situation where more than one wave is present, the longer period wave will travel faster. From Equation II-1-7, it is seen that Equation II-1-8 can be written as

$$C = \frac{gT}{2\pi} \tanh\left(\frac{2\pi d}{L}\right) \tag{II-1-9}$$

(d) The values $2\pi/L$ and $2\pi/T$ are called the *wave number* k and the *wave angular frequency* ω, respectively. From Equation II-1-7 and II-1-9, an expression for wavelength as a function of depth and wave period may be obtained as

$$L = \frac{gT^2}{2\pi} \tanh\left(\frac{2\pi d}{L}\right) = \frac{gT}{\omega} \tanh\left(kd\right) \tag{II-1-10}$$

(e) Use of Equation II-1-10 involves some difficulty since the unknown L appears on both sides of the equation. Tabulated values of d/L and d/L_0 (SPM 1984) where L_0 is the deepwater wavelength may be used to simplify the solution of Equation II-1-10. Eckart (1952) gives an approximate expression for Equation II-1-10, which is correct to within about 10 percent. This expression is given by

$$L \approx \frac{gT^2}{2\pi} \sqrt{\tanh\left(\frac{4\pi^2}{T^2}\frac{d}{g}\right)} \tag{II-1-11}$$

(f) Equation II-1-11 explicitly gives L in terms of wave period T and is sufficiently accurate for many engineering calculations. The maximum error 10 percent occurs when $d/L \approx 1/2$. There are several other approximations for solving Equation II-1-10 (Hunt 1979; Venezian and Demirbilek 1979; Wu and Thornton 1986; Fenton and McKee 1990).

(g) Gravity waves may also be classified by the water depth in which they travel. The following classifications are made according to the magnitude of d/L and the resulting limiting values taken by the function $\tanh(2\pi d/L)$. Note that as the argument of the hyperbolic tangent $kd = 2\pi d/L$ gets large, the $\tanh(kd)$ approaches 1, and for small values of kd, $\tanh(kd) \approx kd$.

(h) Water waves are classified in Table II-1-1 based on the relative depth criterion d/L.

Table II-1-1
Classification of Water Waves

Classification	d/L	kd	tanh (kd)
Deep water	1/2 to ∞	π to ∞	≈1
Transitional	1/20 to 1/2	π/10 to π	tanh (kd)
Shallow water	0 to 1/20	0 to π/10	≈ kd

(i) In deep water, tanh (kd) approaches unity, Equations II-1-7 and II-1-8 reduce to

$$C_0 = \sqrt{\frac{gL_0}{2\pi}} = \frac{L_0}{T} \tag{II-1-12}$$

and Equation II-1-9 becomes

$$C_0 = \frac{gT}{2\pi} \tag{II-1-13}$$

(j) Although *deep water* actually occurs at an infinite depth, *tanh (kd)*, for most practical purposes, approaches unity at a much smaller *d/L*. For a relative depth of one-half (i.e., when the depth is one-half the wavelength), tanh (2π*d/L*) = 0.9964.

(k) When the relative depth *d/L* is greater than one-half, the wave characteristics are virtually independent of depth. Deepwater conditions are indicated by the subscript *0* as in L_o and C_o except that the period *T* remains constant and independent of depth for oscillatory waves, and therefore, the subscript for wave period is omitted (Ippen 1966). In the SI system (System International or metric system of units) where units of meters and seconds are used, the constant *g/2π* is equal to 1.56 m/s^2, and

$$C_0 = \frac{gT}{2\pi} = \frac{9.8}{2\pi}\,T = 1.56\,T\ m/s \tag{II-1-14}$$

and

$$L_0 = \frac{gT^2}{2\pi} = \frac{9.8}{2\pi}\,T^2 = 1.56\,T^2\ m \tag{II-1-15}$$

(l) If units of feet and seconds are specified, the constant *g/2π* is equal to 5.12 ft/s^2, and

$$C_0 = \frac{gT}{2\pi} = 5.12\,T\ ft/s \tag{II-1-16}$$

and

$$L_0 = \frac{gT^2}{2\pi} = 5.12\,T^2\ ft \tag{II-1-17}$$

(m) If Equations II-1-14 and II-1-15 are used to compute wave celerity when the relative depth is *d/L* = 0.25, the resulting error will be about 9 percent. It is evident that a relative depth of 0.5 is a satisfactory boundary separating deepwater waves from waves in water of *transitional depth*. If a wave is traveling in *transitional* depths, Equations II-1-8 and II-1-9 must be used without simplification. As a rule of thumb, Equation II-1-8 and II-1-9 must be used when the relative depth is between 0.5 and 0.04.

(n) When the relative water depth becomes shallow, i.e., $2\pi d/L < 1/4$ or $d/L < 1/25$, Equation II-1-8 can be simplified to

$$C = \sqrt{gd} \tag{II-1-18}$$

(o) Waves sufficiently long such that Equation II-1-18 may be applied are termed long waves. This relation is attributed to Lagrange. Thus, when a wave travels in shallow water, wave celerity depends only on water depth.

(p) In summary, as a wind wave passes from deep water to the beach its speed and length are first only a function of its period (or frequency); then as the depth becomes shallower relative to its length, the length and speed are dependent upon both depth and period; and finally the wave reaches a point where its length and speed are dependent only on depth (and not frequency).

(3) The sinusoidal wave profile. The equation describing the free surface as a function of time t and horizontal distance x for a simple sinusoidal wave can be shown to be

$$\eta = a \cos(kx - \omega t) = \frac{H}{2} \cos\left(\frac{2\pi x}{L} - \frac{2\pi t}{T}\right) = a \cos\theta \tag{II-1-19}$$

where η is the elevation of the water surface relative to the SWL, and $H/2$ is one-half the wave height equal to the wave amplitude a. This expression represents a periodic, sinusoidal, progressive wave traveling in the positive x-direction. For a wave moving in the negative x-direction, the minus sign before $2\pi t/T$ is replaced with a plus sign. When $\theta = (2\pi x/L - 2\pi t/T)$ equals 0, $\pi/2$, π, $3\pi/2$, the corresponding values of η are $H/2$, 0, $-H/2$, and 0, respectively (Figure II-1-1).

(4) Some useful functions.

(a) Dividing Equation II-1-9 by Equation II-1-13, and Equation II-1-10 by Equation II-1-15 yields,

$$\frac{C}{C_0} = \frac{L}{L_0} = \tanh\left(\frac{2\pi d}{L}\right) = \tanh kd \tag{II-1-20}$$

(b) If both sides of Equation II-1-20 are multiplied by d/L, it becomes

$$\frac{d}{L_0} = \frac{d}{L} \tanh\left(\frac{2\pi d}{L}\right) = \frac{d}{L} \tanh kd \tag{II-1-21}$$

(c) The terms d/L_o and d/L and other useful functions such as $kd = 2\pi d/L$ and $\tanh(kd)$ have been tabulated by Wiegel (1954) as a function of d/L_o (see also SPM 1984, Appendix C, Tables C-1 and C-2). These functions simplify the solution of wave problems described by the linear theory and are summarized in Figure II-1-5. An example problem illustrating the use of linear wave theory equations and the figures and tables mentioned follows.

EXAMPLE PROBLEM II-1-1

FIND:

The wave celerities C and lengths L corresponding to depths d = 200 meters (656 ft) and d = 3 m (9.8 ft).

GIVEN:

A wave with a period T = 10 seconds is propagated shoreward over a uniformly sloping shelf from a depth d = 200 m (656 ft) to a depth d = 3 m (9.8 ft).

SOLUTION:

Using Equation II-1-15,

$$L_0 = \frac{gT^2}{2\pi} = \frac{9.8\ T^2}{2\pi} = 1.56\ T^2\ m\ (5.12\ T^2\ ft)$$

$$L_0 = 1.56T^2 = 1.56(10)^2 = 156\ m\ (512\ ft)$$

For d = 200 m

$$\frac{d}{L_0} = \frac{200}{156} = 1.2821$$

Note that for values of

$$\frac{d}{L_0} > 1.0$$

$$\frac{d}{L_0} = \frac{d}{L}$$

therefore,

$$L = L_0 = 156\ m\ (512\ ft)\ (deepwater\ wave,\ since \frac{d}{L} > \frac{1}{2})$$

which is in agreement with Figure II-1-5.

By Equation II-1-7

$$C = \frac{L}{T} = \frac{156}{T}$$

$$C = \frac{156}{10} = 15.6\ m/s\ (51.2\ ft/s)$$

For d = 3 m

$$\frac{d}{L_0} = \frac{3}{156} = 0.0192$$

Example Problem II-1-1 (Continued)

Example Problem II-1-1 (Concluded)

By trial-and-error solution (Equation II-1-21) with d/L_o it is found that

$$\frac{d}{L} = 0.05641$$

hence

$$L = \frac{3}{0.05641} = 53.2 \ m \ (174 \ ft) \left(\text{transitional depth, since} \frac{1}{25} < \frac{\mathrm{d}}{\mathrm{L}} < \frac{1}{2} \right)$$

$$C = \frac{L}{T} = \frac{53.2}{10} = 5.32 \ m/s \ (17.4 \ ft/s)$$

An approximate value of L can also be found by using Equation II-1-11

$$L \approx \frac{gT^2}{2\pi} \sqrt{\tanh\left(\frac{4\pi^2}{T^2} \frac{d}{g} \right)}$$

which can be written in terms of L_o as

$$L \simeq L_0 \sqrt{\tanh\left(\frac{2\pi d}{L_0} \right)}$$

therefore

$$L \simeq 156 \sqrt{\tanh\left(\frac{2\pi(3)}{156} \right)}$$

$$L \simeq 156 \sqrt{\tanh(0.1208)}$$

$$L \simeq 156 \sqrt{0.1202} = 54.1 \ m \ (177.5 \ ft)$$

which compares with L = 53.3 m obtained using Equations II-1-8, II-1-9, or II-1-21. The error in this case is 1.5 percent. Note that Figure II-1-5 or Plate C-1 (SPM 1984) could also have been used to determine d/L.

(5) Local fluid velocities and accelerations.

(a) In wave force studies, the local fluid velocities and accelerations for various values of *z* and *t* during the passage of a wave must often be found. The horizontal component *u* and the vertical component *w* of the local fluid velocity are given by the following equations (with θ, x, and t as defined in Figure II-1-1):

$$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta \tag{II-1-22}$$

$$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta \tag{II-1-23}$$

(b) These equations express the local fluid velocity components any distance $(z + d)$ above the bottom. The velocities are periodic in both *x* and *t*. For a given value of the phase angle $\theta = (2\pi x/L - 2\pi t/T)$, the hyperbolic functions *cosh* and *sinh*, as functions of *z* result in an approximate exponential decay of the magnitude of velocity components with increasing distance below the free surface. The maximum positive horizontal velocity occurs when $\theta = 0$, 2π, etc., while the maximum horizontal velocity in the negative direction occurs when $\theta = \pi$, 3π, etc. On the other hand, the maximum positive vertical velocity occurs when $\theta = \pi/2$, $5\pi/2$, etc., and the maximum vertical velocity in the negative direction occurs when $\theta = 3\pi/2$, $7\pi/2$, etc. Fluid particle velocities under a wave train are shown in Figure II-1-2.

(c) The local fluid particle accelerations are obtained from Equations II-1-22 and II-1-23 by differentiating each equation with respect to *t*. Thus,

$$\alpha_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta = \frac{\partial u}{\partial t} \tag{II-1-24}$$

$$\alpha_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta = \frac{\partial w}{\partial t} \tag{II-1-25}$$

(d) Positive and negative values of the horizontal and vertical fluid accelerations for various values of θ are shown in Figure II-1-2.

(e) Figure II-1-2, a sketch of the local fluid motion, indicates that the fluid under the crest moves in the direction of wave propagation and returns during passage of the trough. Linear theory does not predict any net mass transport; hence, the sketch shows only an oscillatory fluid motion. Figure II-1-3 depicts profiles of the surface elevation, particle velocities, and accelerations by the linear wave theory. The following problem illustrates the computations required to determine local fluid velocities and accelerations resulting from wave motions.

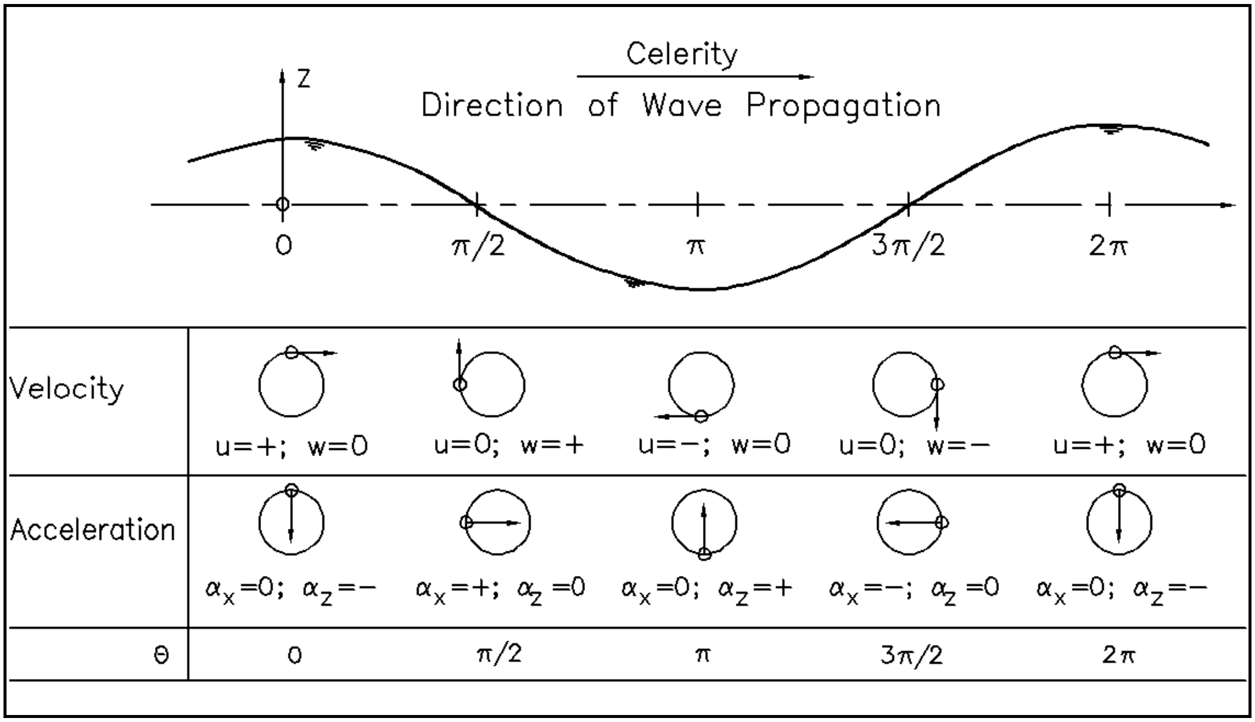

Figure II-1-2. Local fluid velocities and accelerations

(6) Water particle displacements.

(a) Another important aspect of linear wave theory deals with the displacement of individual water particles within the wave. Water particles generally move in elliptical paths in shallow or transitional depth water and in circular paths in deep water (Figure II-1-4). If the mean particle position is considered to be at the center of the ellipse or circle, then vertical particle displacement with respect to the mean position cannot exceed one-half the wave height. Thus, since the wave height is assumed to be small, the displacement of any fluid particle from its mean position must be small. Integration of Equations II-1-22 and II-1-23 gives the horizontal and vertical particle displacements from the mean position, respectively (Figure II-1-4).

(b) Fluid particle displacements are

$$\xi = -\frac{HgT^2}{4\pi L} \frac{\cosh\left(\frac{2\pi(z+d)}{L}\right)}{\cosh\left(\frac{2\pi d}{L}\right)} \sin\,\theta \tag{II-1-26}$$

$$\zeta = +\frac{HgT^2}{4\pi L} \frac{\sinh\left(\frac{2\pi(z+d)}{L}\right)}{\cosh\left(\frac{2\pi d}{L}\right)} \cos\,\theta \tag{II-1-27}$$

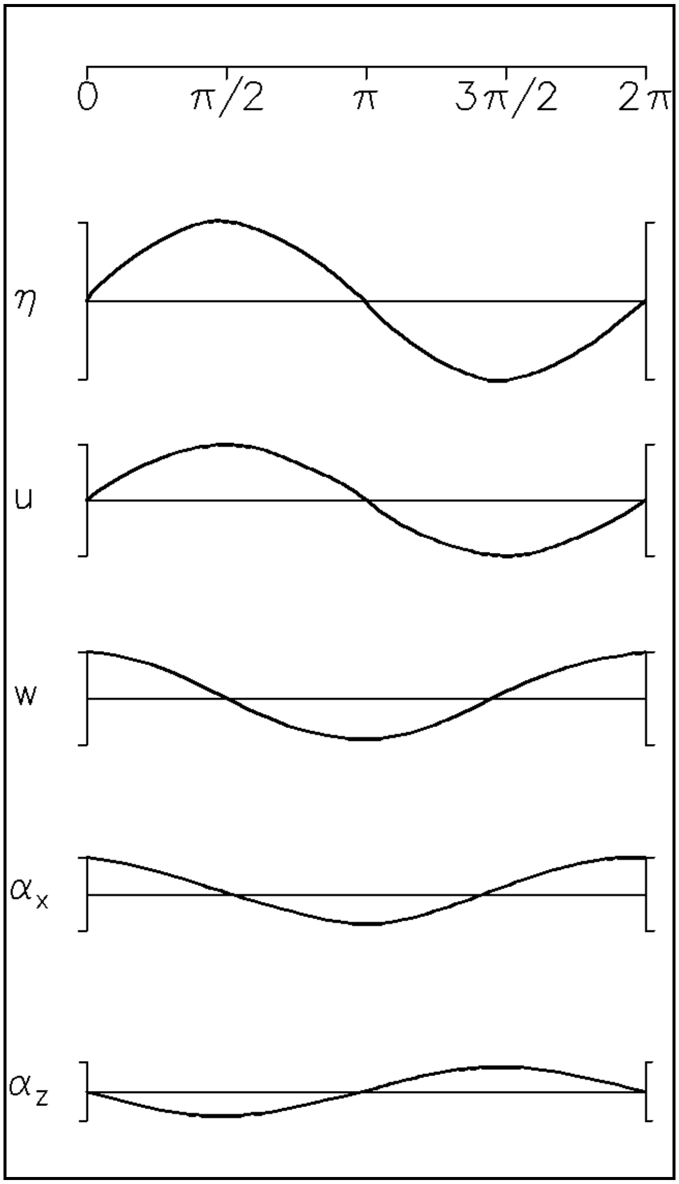

Figure II-1-3. Profiles of particle velocity and acceleration by Airy theory in relation to the surface elevation

where ξ is the horizontal displacement of the water particle from its mean position and ζ is the vertical displacement from its mean position (Figure II-1-4). The above equations can be simplified by using the relationship

$$\left(\frac{2\pi}{T}\right)^2 = \frac{2\pi g}{L}\tanh\frac{2\pi d}{L} \qquad \text{(II-1-28)}$$

EXAMPLE PROBLEM II-1-2

FIND:

The local horizontal and vertical velocities *u* and *w*, and accelerations α_x and α_z at an elevation z = -5 m (or z = -16.4 ft) below the SWL when $\theta = 2\pi x/L - 2\pi t/T = \pi/3$ (or 60^0).

GIVEN:

A wave with a period T = 8 sec, in a water depth d = 15 m (49 ft), and a height H = 5.5 m (18.0 ft).

SOLUTION:

Calculate

$$L_0 = 1.56T^2 = 1.56(8)^2 = 99.8 \; m \; (327 \; ft)$$

$$\frac{d}{L_0} = \frac{15}{99.8} = 0.1503$$

By trial-and-error solution or using Figure II-1-5 for d/L_0 = 0.1503, we find

$$\frac{d}{L} = 0.1835$$

and

$$\cosh\frac{2\pi d}{L} = 1.742$$

hence

$$L = \frac{15}{0.1835} = 81.7 \; m \; (268 \; ft)$$

Evaluation of the constant terms in Equations II-1-22 to II-1-25 gives

$$\frac{HgT}{2L}\,\frac{1}{\cosh(2\pi d/L)} = \frac{5.5\;(9.8)(8)}{2\;(81.7)}\,\frac{1}{1.742} = 1.515$$

$$\frac{Hg\pi}{L}\,\frac{1}{\cosh(2\pi d/L)} = \frac{5.5\;(9.8)(3.1416)}{81.7}\,\frac{1}{1.742} = 1.190$$

Substitution into Equation II-1-22 gives

$$u = 1.515 \; \cosh\left[\frac{2\pi(15 - 5)}{81.7}\right][\cos \; 60^0]$$

$$= 1.515 \; [\cosh(0.7691)] \; (0.500)$$

Example Problem II-1-2 (Continued)

Example Problem II-1-2 (Concluded)

From the above known information, we find

$$\frac{2\pi d}{L} = 0.7691$$

and values of hyperbolic functions become

$$\cosh(0.7691) = 1.3106$$

and

$$\sinh(0.7691) = 0.8472$$

Therefore, fluid particle velocities are

$$u = 1.515(1.1306)(0.500) = 0.99 \; m/s \; (3.26 \; ft/s)$$

$$w = 1.515(0.8472)(0.866) = 1.11 \; m/s \; (3.65 \; ft/s)$$

and fluid particle accelerations are

$$\alpha_x = 1.190(1.3106)(0.866) = 1.35 \; m/s^2 \; (4.43 \; ft/s^2)$$

$$\alpha_z = -1.190(0.8472)(0.500) = -0.50 \; m/s^2 \; (1.65 \; ft/s^2)$$

(c) Thus,

$$\xi = -\frac{H}{2} \frac{\cosh\left(\frac{2\pi(z+d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)} \sin \theta \qquad \text{(II-1-29)}$$

$$\zeta = +\frac{H}{2} \frac{\sinh\left(\frac{2\pi(z+d)}{L}\right)}{\sinh\left(\frac{2\pi d}{L}\right)} \cos \theta \qquad \text{(II-1-30)}$$

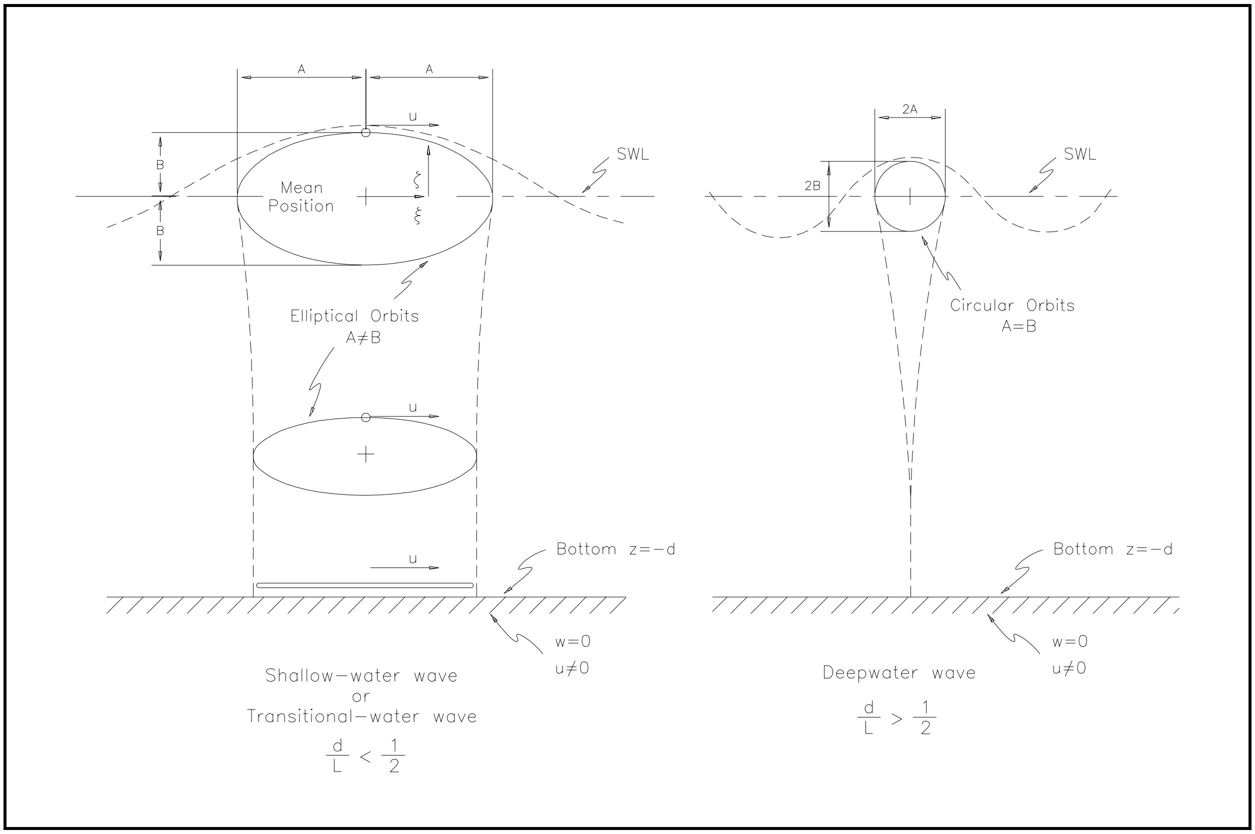

Figure II-1-4. Water particle displacements from mean position for shallow-water and deepwater waves

(d) Writing Equations II-1-29 and II-1-30 in the forms,

$$\sin^2 \theta = \left[\frac{\xi}{a} \frac{\sinh\left(\frac{2\pi d}{L}\right)}{\cosh\left(\frac{2\pi(z+d)}{L}\right)} \right]^2 \qquad \text{(II-1-31)}$$

$$\cos^2 \theta = \left[\frac{\zeta}{a} \frac{\sinh\left(\frac{2\pi d}{L}\right)}{\sinh\left(\frac{2\pi(z + d)}{L}\right)} \right]^2 \qquad \text{(II-1-32)}$$

and adding gives

$$\frac{\xi^2}{A^2} + \frac{\zeta^2}{B^2} = 1 \qquad \text{(II-1-33)}$$

in which *A* and *B* are

$$A = \frac{H}{2} \frac{\cosh\left(\dfrac{2\pi(z + d)}{L}\right)}{\sinh\left(\dfrac{2\pi d}{L}\right)} \tag{II-1-34}$$

$$B = \frac{H}{2} \frac{\sinh\left(\dfrac{2\pi(z + d)}{L}\right)}{\sinh\left(\dfrac{2\pi d}{L}\right)} \tag{II-1-35}$$

(e) Equation II-1-33 is the equation of an ellipse with a major- (horizontal) semi-axis equal to A and a minor (vertical) semi-axis equal to B. The lengths of A and B are measures of the horizontal and vertical displacements of the water particles (see Figure II-1-4). Thus, the water particles are predicted to move in closed orbits by linear wave theory; i.e., a fluid particle returns to its initial position after each wave cycle. Comparing laboratory measurements of particle orbits with this theory shows that particle orbits are not completely closed. This difference between linear theory and observations is due to the mass transport phenomenon, which is discussed later in this chapter. It shows that linear theory is inadequate to explain wave motion completely.

(f) Examination of Equations II-1-34 and II-1-35 shows that for deepwater conditions, A and B are equal and particle paths are circular (Figure II-1-4). These equations become

$$A = B = \frac{H}{2} e^{\left(\frac{2\pi z}{L}\right)} \quad \text{for } \frac{\mathrm{d}}{\mathrm{L}} > \frac{1}{2} \text{ (i.e., deepwater limit)} \tag{II-1-36}$$

(g) For shallow-water conditions (d/L < 1/25), the equations become

$$A = \frac{H}{2} \frac{L}{2\pi d} \tag{II-1-37}$$

and

$$B = \frac{H}{2} \left(1 + \frac{z}{d}\right) \tag{II-1-38}$$

EXAMPLE PROBLEM II-1-3

FIND:

(a) The maximum horizontal and vertical displacement of a water particle from its mean position when $z = 0$ and $z = -d$.

(b) The maximum water particle displacement at an elevation $z = -7.5$ m (-24.6 ft) when the wave is in infinitely deep water.

(c) For the deepwater conditions of (b) above, show that the particle displacements are small relative to the wave height when $z = -L_0/2$.

GIVEN:

A wave in a depth $d = 12$ m (39.4 ft), height $H = 3$ m (9.8 ft), and a period $T = 10$ sec. The corresponding deepwater wave height is $H_0 = 3.13$ m (10.27 ft).

SOLUTION:

(a)

$$L_0 = 1.56T^2 = 1.56(10)^2 = 156\ m\ (512\ ft)$$

$$\frac{d}{L_0} = \frac{12}{156} = 0.0769$$

From hand calculators, we find

$$\sinh\left(\frac{2\pi d}{L}\right) = 0.8306$$

$$\tanh\left(\frac{2\pi d}{L}\right) = 0.6389$$

When z = 0, Equation II-1-34 reduces to

$$A = \frac{H}{2}\frac{1}{\tanh\left(\frac{2\pi d}{L}\right)}$$

and Equation II-1-35 reduces to

$$B = \frac{H}{2}$$

Thus

$$A = \frac{3}{2}\frac{1}{(0.6389)} = 2.35\ m\ (7.70\ ft)$$

$$B = \frac{H}{2} = \frac{3}{2} = 1.5\ m\ (4.92\ ft)$$

Example Problem II-1-3 (Continued)

Example Problem II-1-3 (Concluded)

When z = -d,

$$A = \frac{H}{2 \sinh\left(\frac{2\pi d}{L}\right)} = \frac{3}{2(0.8306)} = 1.81 \ m \ (5.92 \ ft)$$

and B = 0.

(b) With H_0 = 3.13 m and z = -7.5 m (-24.6 ft), evaluate the exponent of e for use in Equation II-1-36, noting that $L = L_0$,

$$\frac{2\pi z}{L} = \frac{2\pi(-7.5)}{156} = -0.302$$

thus,

$$e^{-0.302} = 0.739$$

Therefore,

$$A = B = \frac{H_0}{2} e^{\left(\frac{2\pi z}{L}\right)} = \frac{3.13}{2}(0.739) = 1.16 \ m \ (3.79 \ ft)$$

The maximum displacement or diameter of the orbit circle would be 2(1.16) = 2.32 m (7.61 ft) when z = -7.5 m.

(c) At a depth corresponding to the half wavelength from the MWL, we have

$$z = -\frac{L_0}{2} = \frac{-156}{2} = -78.0 \ m \ (255.9 \ ft)$$

$$\frac{2\pi z}{L} = \frac{2\pi(-78)}{156} = -3.142$$

Therefore

$$e^{-3.142} = 0.043$$

and

$$A = B = \frac{H_0}{2} e^{\left(\frac{2\pi z}{L}\right)} = \frac{3.13}{2} (0.043) = 0.067 \ m \ (0.221 \ ft)$$

Thus, the maximum displacement of the particle is 0.067 m, which is small when compared with the deepwater height, H_0 = 3.13 m (10.45 ft).

(h) Thus, in deep water, the water particle orbits are circular as indicated by Equation II-1-36 (see Figure II-1-4). Equations II-1-37 and II-1-38 show that in transitional and shallow water, the orbits are elliptical. The more shallow the water, the flatter the ellipse. The amplitude of the water particle displacement decreases exponentially with depth and in deepwater regions becomes small relative to the wave height at a depth equal to one-half the wavelength below the free surface; i.e., when $z = L_0/2$.

(i) Water particle displacements and orbits based on linear theory are illustrated in Figure II-1-4. For shallow regions, horizontal particle displacement near the bottom can be large. In fact, this is apparent in offshore regions seaward of the breaker zone where wave action and turbulence lift bottom sediments into suspension. The vertical displacement of water particles varies from a minimum of zero at the bottom to a maximum equal to one-half the wave height at the surface.

(7) Subsurface pressure.

(a) Subsurface pressure under a wave is the sum of two contributing components, dynamic and static pressures, and is given by

$$p' = \frac{\rho g H \cosh\left[\frac{2\pi(z+d)}{L}\right]}{2\cosh\left(\frac{2\pi d}{L}\right)} \cos\theta - \rho g z + p_a \tag{II-1-39}$$

where p' is the total or absolute pressure, p_a is the atmospheric pressure, and ρ is the mass density of water (for salt water, $\rho = 1{,}025\ \mathrm{kg/m^3}$ or $2.0\ \mathrm{slugs/ft^3}$, for fresh water, $\rho = 1{,}000\ \mathrm{kg/m^3}$ or $1.94\ \mathrm{slugs/ft^3}$). The first term of Equation II-1-39 represents a dynamic component due to acceleration, while the second term is the static component of pressure. For convenience, the pressure is usually taken as the gauge pressure defined as

$$p = p' - p_a = \frac{\rho g H \cosh\left[\frac{2\pi(z+d)}{L}\right]}{2\cosh\left(\frac{2\pi d}{L}\right)} \cos\theta - \rho g z \tag{II-1-40}$$

(b) Equation II-1-40 can be written as

$$p = \rho g \eta \frac{\cosh\left[\frac{2\pi(z+d)}{L}\right]}{\cosh\left(\frac{2\pi d}{L}\right)} - \rho g z \tag{II-1-41}$$

since

$$\eta = \frac{H}{2}\cos\left(\frac{2\pi x}{L} - \frac{2\pi t}{T}\right) = \frac{H}{2}\cos\theta \tag{II-1-42}$$

(c) The ratio

$$K_z = \frac{\cosh\left[\frac{2\pi(z + d)}{L}\right]}{\cosh\left(\frac{2\pi d}{L}\right)} \tag{II-1-43}$$

is termed *the pressure response factor*. Hence, Equation II-1-41 can be written as

$$p = \rho g(\eta K_z - z) \tag{II-1-44}$$

(d) The pressure response factor K for the pressure at the bottom when z = -d,

$$K_z = K = \frac{1}{\cosh\left(\frac{2\pi d}{L}\right)} \tag{II-1-45}$$

is presented as function of d/L_0 in the tables (SPM 1984); see also Figure II-1-5. This figure is a convenient graphic means to determine intermediate and shallow-water values of the bottom pressure response factor K, the ratio C/C_0 ($=L/L_0 = k_0/k$), and a number of other variables commonly occurring in water wave calculations.

(e) It is often necessary to determine the height of surface waves based on subsurface measurements of pressure. For this purpose, it is convenient to rewrite Equation II-1-44 as

$$\eta = \frac{N(p + \rho g z)}{\rho g K_z} \tag{II-1-46}$$

where z is the depth below the SWL of the pressure gauge, and N a correction factor equal to unity if the linear theory applies.

(f) Chakrabarti (1987) presents measurements that correlate measured dynamic pressure in the water column (s in his notation is the elevation above the seabed) with linear wave theory. These laboratory measurements include a number of water depths, wave periods, and wave heights. The best agreement between the theory and these measurements occurs in deep water. Shallow-water pressure measurements for steep water waves deviate significantly from the linear wave theory predictions. The example problem hereafter illustrates the use of pertinent equations for finding wave heights from pressure measurements based on linear theory.

(8) Group velocity.

(a) It is desirable to know how fast wave energy is moving. One way to determine this is to look at the speed of wave groups that represents propagation of wave energy in space and time. The speed a group of

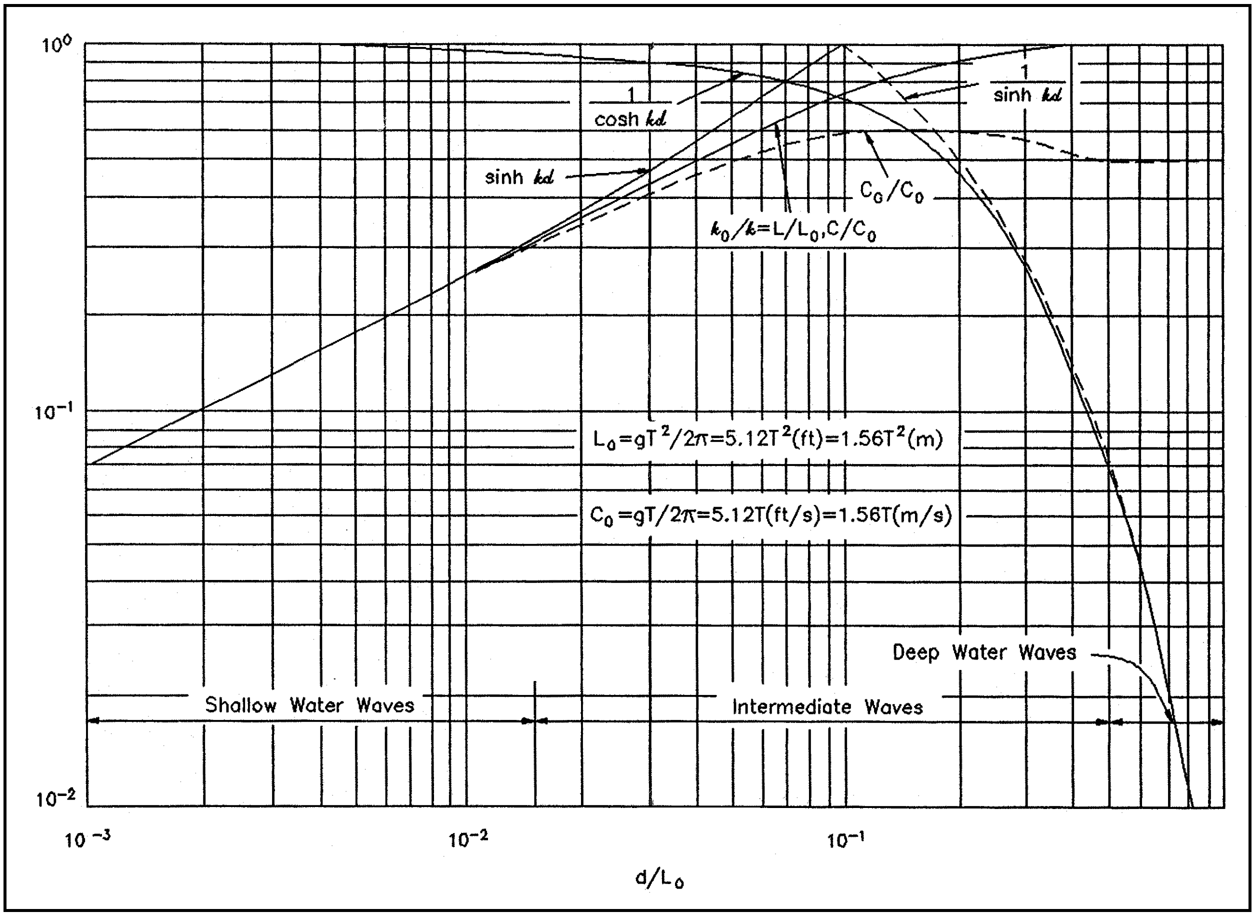

Figure II-1-5. Variation of wave parameters with d/L_0 (Dean and Dalrymple 1991)

waves or a wave train travels is generally not identical to the speed with which individual waves within the group travel. The group speed is termed the *group velocity* C_g; the individual wave speed is the *phase velocity* or *wave celerity* given by Equations II-1-8 or II-1-9. For waves propagating in deep or transitional water with gravity as the primary restoring force, the group velocity will be less than the phase velocity. For those waves, propagated primarily under the influence of surface tension (i.e., capillary waves), the group velocity may exceed the velocity of an individual wave.

(b) The concept of group velocity can be described by considering the interaction of two sinusoidal wave trains moving in the same direction with slightly different wavelengths and periods. The equation of the water surface is given by

$$\eta = \eta_1 + \eta_2 = \frac{H}{2}\cos\left(\frac{2\pi x}{L_1} - \frac{2\pi t}{T_1}\right) + \frac{H}{2}\cos\left(\frac{2\pi x}{L_2} - \frac{2\pi t}{T_2}\right) \qquad \text{(II-1-47)}$$

where η_1 and η_2 are the two components. They may be summed since superposition of solutions is permissible when the linear wave theory is used. For simplicity, the heights of both wave components have been assumed equal. Since the wavelengths of the two component waves, L_1 and L_2, have been assumed slightly different for some values of x at a given time, the two components will be in phase and the wave height observed will be 2H; for some other values of x, the two waves will be completely out of phase and

EXAMPLE PROBLEM II-1-4

FIND:

The height of the wave H assuming that linear theory applies and the average frequency corresponds to the average wave amplitude.

GIVEN:

An average maximum pressure p = 124 kilonewtons per square meter is measured by a subsurface pressure gauge located in salt water 0.6 meter (1.97 ft) above the bed in depth d = 12 m (39 ft). The average frequency f = 0.06666 cycles per second (Hertz).

SOLUTION:

$$T = \frac{1}{f} = \frac{1}{(0.0666)} \approx 15 \ s$$

$$L_0 = 1.56T^2 = 1.56(15)^2 = 351 \ m \ (1152 \ ft)$$

$$\frac{d}{L_0} = \frac{12}{351} \approx 0.0342$$

From Figure II-1-5, entering with d/L_0,

$$\frac{d}{L} = 0.07651$$

hence,

$$L = \frac{12}{(0.07651)} = 156.8 \ m \ (515 \ ft)$$

and

$$\cosh\left(\frac{2\pi d}{L}\right) = 1.1178$$

Therefore, from Equation II-1-43

$$K_z = \frac{\cosh\left[\dfrac{2\pi(z+d)}{L}\right]}{\cosh\left(\dfrac{2\pi d}{L}\right)} = \frac{\cosh\left[\dfrac{2\pi(-11.4+12)}{156.8}\right]}{1.1178} = 0.8949$$

Since η = a = H/2 when the pressure is maximum (under the wave crest), and N = 1.0 since linear theory is assumed valid,

$$\frac{H}{2} = \frac{N(p + \rho g z)}{\rho g K_z} = \frac{1.0 \ [124 + (10.06)\ (-11.4)]}{(10.06)\ (0.8949)} = 1.04 \ m \ (3.44 \ ft)$$

Therefore,

$$H = 2(1.04) = 2.08 \ m \ (6.3 \ ft)$$

Note that the value of K in Figure II-1-5 or SPM (1984) could not be used since the pressure was not measured at the bottom.

the resultant wave height will be zero. The surface profile made up of the sum of the two sinusoidal waves is given by Equation II-1-47 and is shown in Figure II-1-6. The waves shown in Figure II-1-6 appear to be traveling in groups described by the equation of the envelope curves

$$\eta_{envelope} = \pm H \cos\left[\pi\left(\frac{L_2 - L_1}{L_1 L_2}\right)x - \pi\left(\frac{T_2 - T_1}{T_1 T_2}\right)t\right] \tag{II-1-48}$$

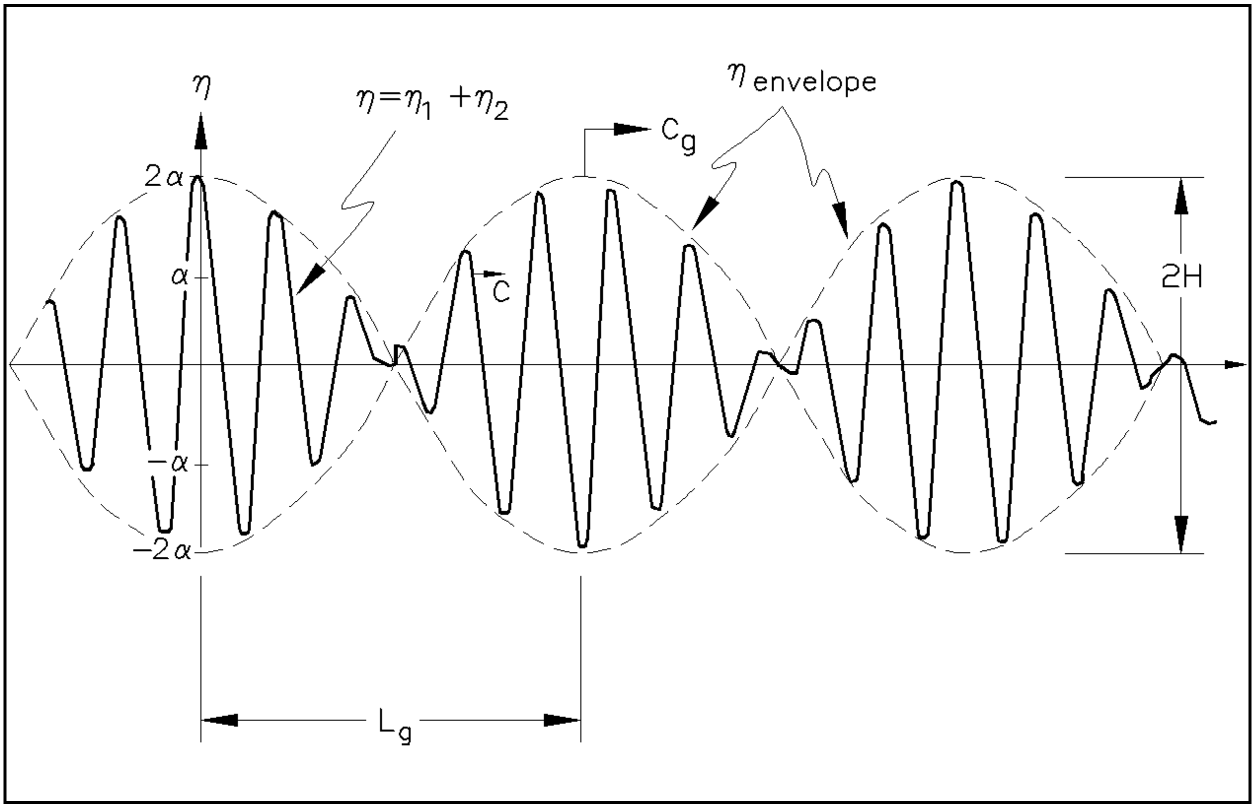

Figure II-1-6. Characteristics of a wave group formed by the addition of sinusoids with different periods

(c) It is the speed of these groups (i.e., the velocity of propagation of the envelope curves) defined in Equation II-1-48 that represents the group velocity. The limiting speed of the wave groups as they become large (i.e., as the wavelength L_1 approaches L_2 and consequently the wave period T_1 approaches T_2) is the group velocity and can be shown to be equal to

$$C_g = \frac{1}{2}\frac{L}{T}\left[1 + \frac{\frac{4\pi d}{L}}{\sinh\left(\frac{4\pi d}{L}\right)}\right] = nC \tag{II-1-49}$$

where

$$n = \frac{1}{2}\left[1 + \frac{\frac{4\pi d}{L}}{\sinh\left(\frac{4\pi d}{L}\right)}\right] \tag{II-1-50}$$

(d) In deep water, the term (4πd/L)/sinh(4πd/L) is approximately zero and $n = 1/2$, giving

$$C_{g_0} = \frac{1}{2}\frac{L_0}{T} = \frac{1}{2}C_0 \text{ (deep water)} \tag{II-1-51}$$

or the group velocity is one-half the phase velocity.

(e) In shallow water, sinh(4πd/L ≈ 4πd/L) and

$$C_{g_s} = \frac{L}{T} = C \approx \sqrt{gd} \text{ (shallow water)} \tag{II-1-52}$$

hence, the group and phase velocities are equal. Thus, in shallow water, because wave celerity is determined by the depth, all component waves in a wave train will travel at the same speed precluding the alternate reinforcing and canceling of components. In deep and transitional water, wave celerity depends on wavelength; hence, slightly longer waves travel slightly faster and produce the small phase differences resulting in wave groups. These waves are said to be *dispersive* or propagating in a *dispersive medium*; i.e., in a medium where their celerity is dependent on wavelength.

(f) The variation of the ratios of group and phase velocities to the deepwater phase velocity C_g/C_0 and C/C_0, respectively are given as a function of the depth relative to the deep water wavelength d/L_0 in Figure II-1-7. The two curves merge together for small values of depth and C_g reaches a maximum before tending asymptotically toward $C/2$.

(g) Outside of shallow water, the phase velocity of gravity waves is greater than the group velocity. An observer that follows a group of waves at group velocity will see waves that originate at the rear of the group move forward through the group traveling at the phase velocity and disappear at the front of the wave group.

(h) Group velocity is important because it is with this velocity that wave energy is propagated. Although mathematically the group velocity can be shown rigorously from the interference of two or more waves (Lamb 1945), the physical significance is not as obvious as it is in the method based on the consideration of wave energy. Therefore an additional explanation of group velocity is provided on wave energy and energy transmission.

(9) Wave energy and power.

(a) The total energy of a wave system is the sum of its kinetic energy and its potential energy. The kinetic energy is that part of the total energy due to water particle velocities associated with wave motion. The kinetic energy per unit length of wave crest for a wave defined with the linear theory can be found from

$$\overline{E}_k = \int_x^{x+L}\int_{-d}^{\eta} \rho\frac{u^2 + w^2}{2}\, dz\, dx \tag{II-1-53}$$

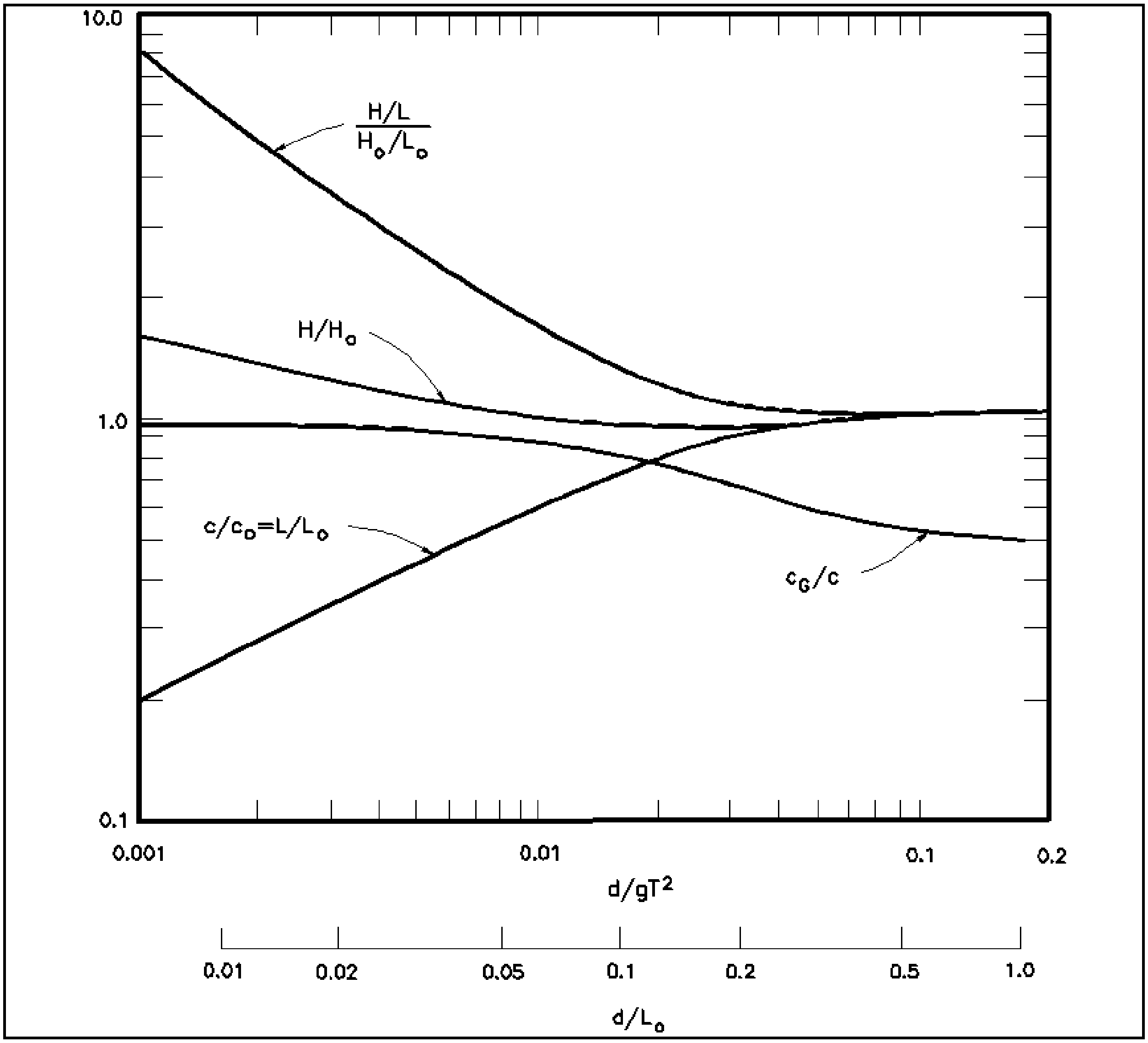

Figure II-1-7. Variation of the ratios of group and phase velocities to deepwater phase speed using linear theory (Sarpkaya and Isaacson 1981)

which, upon integration, gives

$$\overline{E}_k = \frac{1}{16} \rho g H^2 L \tag{II-1-54}$$

(b) Potential energy is that part of the energy resulting from part of the fluid mass being above the trough: the wave crest. The potential energy per unit length of wave crest for a linear wave is given by

$$\overline{E}_p = \int_x^{x+L} \rho g \left[\frac{(\eta + d)^2}{2} - \frac{d^2}{2} \right] dx \tag{II-1-55}$$

which, upon integration, gives

$$\overline{E}_p = \frac{1}{16} \rho \ g \ H^2 \ L \tag{II-1-56}$$

(c) According to the Airy theory, if the potential energy is determined relative to SWL, and all waves are propagated in the same direction, potential and kinetic energy components are equal, and the total wave energy in one wavelength per unit crest width is given by

$$E = E_k + E_p = \frac{\rho g H^2 L}{16} + \frac{\rho g H^2 L}{16} = \frac{\rho g H^2 L}{8} \tag{II-1-57}$$

where subscripts k and p refer to kinetic and potential energies. Total average wave energy per unit surface area, termed the *specific energy* or *energy density*, is given by

$$\overline{E} = \frac{E}{L} = \frac{\rho g H^2}{8} \tag{II-1-58}$$

(d) *Wave energy flux* is the rate at which energy is transmitted in the direction of wave propagation across a vertical plan perpendicular to the direction of wave advance and extending down the entire depth. Assuming linear theory holds, the average energy flux per unit wave crest width transmitted across a vertical plane perpendicular to the direction of wave advance is

$$\overline{P} = \frac{1}{T} \int_t^{t+r} \int_{-d}^{\eta} p \ u \ dz \ dt \tag{II-1-59}$$

which, upon integration, gives

$$\overline{P} = \overline{E} n C = \overline{E} C_g \tag{II-1-60}$$

where $\overline{P}$ is frequently called *wave power*, and the variable n has been defined earlier in Equation II-1-50.

(e) If a vertical plane is taken other than perpendicular to the direction of wave advance, $\overline{P} = E \ C_g \ \sin \theta$, where θ is the angle between the plane across which the energy is being transmitted and the direction of wave advance.

(f) For deep and shallow water, Equation II-1-60 becomes

$$\overline{P}_0 = \frac{1}{2} \overline{E}_0 \ C_o \ (\text{deep water}) \tag{II-1-61}$$

$$\overline{P} = \overline{E} C_g = \overline{E} C \ (\text{shallow water}) \tag{II-1-62}$$

(g) An energy balance for a region through which waves are passing will reveal that, for steady state, the amount of energy entering the region will equal the amount leaving the region provided no energy is added or removed. Therefore, when the waves are moving so that their crests are parallel to the bottom contours

$$\overline{E}_0 \ n_0 \ C_0 = \overline{E} n C \tag{II-1-63}$$

or since

$$n_0 = \frac{1}{2} \tag{II-1-64}$$

$$\frac{1}{2}\,\bar{E}_0\;C_0 = \bar{E}nC \tag{II-1-65}$$

(h) When the wave crests are not parallel to the bottom contours, some parts of the wave will be traveling at different speeds and the wave will be refracted; in this case Equation II-1-65 does not apply (see Parts II-3 and II-4). The rate of energy transmission is important for coastal design, and it requires knowledge of C_g to determine how fast waves move toward shore. The mean rate of energy transmission associated with waves propagating into an area of calm water provides a different physical description of the concept of group velocity.

(i) Equation II-1-65 establishes a relationship between the ratio of the wave height at some arbitrary depth and the deepwater wave height. This ratio, known as the *shoaling coefficient* (see Part II-3 for detail derivation), is dependent on the wave steepness. The variation of shoaling coefficient with wave steepness as a function of relative water depth d/L_0 is shown in Figure II-1-8. Wave shoaling and other related nearshore processes are described in detail in Parts II-3 and II-4.

(10) Summary of linear wave theory.

(a) Equations describing water surface profile particle velocities, particle accelerations, and particle displacements for linear (Airy) theory are summarized in Figure II-1-9. The Corps of Engineers' microcomputer package of computer programs (ACES; Leenknecht et al. 1992) include several software applications for calculating the linear wave theory and associated parameters. Detailed descriptions of the ACES and CMS software to the linear wave theory may be found in the ACES and CMS documentation.

(b) Other wave phenomena can be explained using linear wave theory. For example, observed decreases and increases in the mean water level, termed wave setdown and wave setup, are in essence nonlinear quantities since they are proportional to wave height squared. These nonlinear quantities may be explained using the concept of radiation stresses obtained from linear theory. Maximum wave setdown occurs just seaward of the breaker line. Wave setup occurs between the breaker line and the shoreline and can increase the mean water level significantly. Wave setdown and setup and their estimation are discussed in Part II-4.

(c) *Radiation stresses* are the forces per unit area that arise because of the excess momentum flux due to the presence of waves. In simple terms, there is more momentum flow in the direction of wave advance because the velocity U is in the direction of wave propagation under the wave crest when the instantaneous water surface is high (wave crest) and in the opposite direction when the water surface is low (wave trough). Also, the pressure stress acting under the wave crest is greater than the pressure stress under the wave trough leading to a net stress over a wave period. Radiation stresses arise because of the finite amplitude (height) of the waves. Interestingly, small-amplitude (linear) wave theory can be used to reasonably approximate radiation stresses and explain effects such as wave set down, wave setup, and the generation of longshore currents.

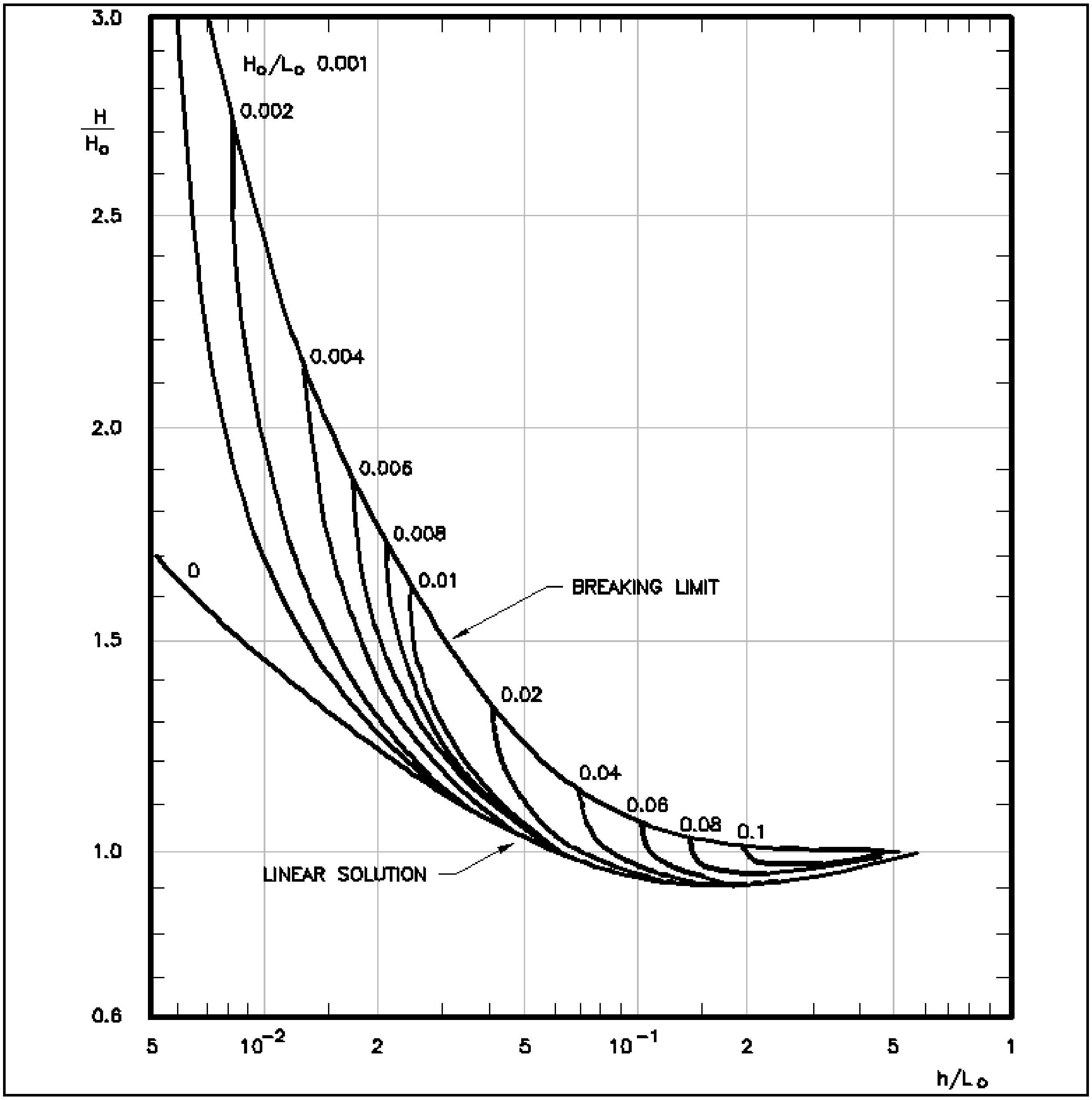

Figure II-1-8. Variation of shoaling coefficient with wave steepness (Sakai and Battjes 1980)

d. Nonlinear wave theories.

(1) Introduction.

(a) Linear waves as well as finite-amplitude waves may be described by specifying two dimensionless parameters, the wave steepness *H/L* and the relative water depth *d/L*. The relative water depth has been discussed extensively earlier in this chapter with regard to linear waves. The *Relative depth* determines whether waves are dispersive or nondispersive and whether the celerity, length, and height are influenced by water depth. *Wave steepness* is a measure of how large a wave is relative to its height and whether the linear wave assumption is valid. Large values of the wave steepness suggest that the small-amplitude

Relative Depth	Shallow Water $\frac{d}{L} < \frac{1}{25}$	Transitional Water $\frac{1}{25} < \frac{d}{L} < \frac{1}{2}$	Deep Water $\frac{d}{L} < \frac{1}{2}$
1. Wave profile	Same As >	$\eta = \frac{H}{2}\cos\left[\frac{2\pi x}{L} - \frac{2\pi t}{T}\right] = \frac{H}{2}\cos\theta$	< Same As
2. Wave celerity	$C = \frac{L}{T} = \sqrt{gd}$	$C = \frac{L}{T} = \frac{gT}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$C = C_0 = \frac{L}{T} = \frac{gT}{2\pi}$
3. Wavelength	$L = T\sqrt{gd} = CT$	$L = \frac{gT^2}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$	$L = L_0 = \frac{gT^2}{2\pi} = C_0 T$
4. Group velocity	$C_g = C = \sqrt{gd}$	$C_g = nC = \frac{1}{2}\left[1 + \frac{4\pi d/L}{\sinh(4\pi d/L)}\right]C$	$C_g = \frac{1}{2}C = \frac{gT}{4\pi}$
5. Water particle velocity			
(a) Horizontal	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\cos\theta$	$u = \frac{H}{2}\frac{gT}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$u = \frac{\pi H}{T}e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
(b) Vertical	$w = \frac{H\pi}{T}\left(1+\frac{z}{d}\right)\sin\theta$	$w = \frac{H}{2}\frac{gT}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$w = \frac{\pi H}{T}e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$
6. Water particle accelerations			
(a) Horizontal	$a_x = \frac{H\pi}{T}\sqrt{\frac{g}{d}}\sin\theta$	$a_x = \frac{g\pi H}{L}\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\sin\theta$	$a_x = 2H\left(\frac{\pi}{T}\right)^2 e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$
(b) Vertical	$a_z = -2H\left(\frac{\pi}{T}\right)^2\left(1+\frac{z}{d}\right)\cos\theta$	$a_z = -\frac{g\pi H}{L}\frac{\sinh[2\pi(z+d)/L]}{\cosh(2\pi d/L)}\cos\theta$	$a_z = -2H\left(\frac{\pi}{T}\right)^2 e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
7. Water particle displacements			
(a) Horizontal	$\xi = -\frac{HT}{4\pi}\sqrt{\frac{g}{d}}\sin\theta$	$\xi = -\frac{H}{2}\frac{\cosh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\sin\theta$	$\xi = -\frac{H}{2}e^{\left(\frac{2\pi z}{L}\right)}\sin\theta$
(b) Vertical	$\zeta = \frac{H}{2}\left(1+\frac{z}{d}\right)\cos\theta$	$\zeta = \frac{H}{2}\frac{\sinh[2\pi(z+d)/L]}{\sinh(2\pi d/L)}\cos\theta$	$\zeta = \frac{H}{2}e^{\left(\frac{2\pi z}{L}\right)}\cos\theta$
8. Subsurface pressure	$p = \rho g(\eta - z)$	$p = \rho g\eta\frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} - \rho g z$	$p = \rho g\eta\, e^{\left(\frac{2\pi z}{L}\right)} - \rho g z$

Figure II-1-9. Summary of linear (Airy) wave theory - wave characteristics

assumption may be questionable. A third dimensionless parameter, which may be used to replace either the wave steepness or relative water depth, may be defined as the ratio of wave steepness to relative water depth. Thus,

$$\frac{H/L}{d/L} = \frac{H}{d} \qquad \text{(II-1-66)}$$

which is termed the *relative wave height*. Like the wave steepness, large values of the relative wave height indicate that the small-amplitude assumption may not be valid. A fourth dimensionless parameter often used to assess the relevance of various wave theories is termed the *Ursell number*. The Ursell number is given by

$$U_R = \left(\frac{L}{d}\right)^2 \frac{H}{d} = \frac{L^2 H}{d^3} \qquad \text{(II-1-67)}$$

(b) The value of the Ursell number is often used to select a wave theory to describe a wave with given L and H (or T and H) in a given water depth d. High values of U_R indicate large, finite-amplitude, long waves in shallow water that may necessitate the use of nonlinear wave theory, to be discussed next.

(c) The linear or small-amplitude wave theory described in the preceding sections provides a useful first approximation to the wave motion. Ocean waves are generally not small in amplitude. In fact, from an engineering point of view it is usually the large waves that are of interest since they result in the largest forces and greatest sediment movement. In order to approach the complete solution of ocean waves more closely, a perturbation solution using successive approximations may be developed to improve the linear theory solution of the hydrodynamic equations for gravity waves. Each order wave theory in the perturbation expansion serves as a correction and the net result is often a better agreement between theoretical and observed waves. The extended theories can also describe phenomena such as *mass transport* where there is a small net forward movement of the water during the passage of a wave. These higher-order or extended solutions for gravity waves are often called *nonlinear wave theories*.

(d) Development of the nonlinear wave theories has evolved for a better description of surface gravity waves. These include *cnoidal*, *solitary*, and *Stokes* theories. However, the development of a Fourier-series approximation by Fenton in recent years has superseded the previous historical developments. Since earlier theories are still frequently referenced, these will first be summarized in this section, but Fenton's theory is recommended for regular waves in all coastal applications.

(2) Stokes finite-amplitude wave theory.

(a) Since the pioneering work of Stokes (1847, 1880) most extension studies (De 1955; Bretschneider 1960; Skjelbreia and Hendrickson 1961; Laitone 1960, 1962, 1965; Chappelear 1962; Fenton 1985) in wave perturbation theory have assumed the wave slope ka is small where k is the wave number and a the amplitude of the wave. The perturbation solution, developed as a power series in terms of $\varepsilon = ka$, is expected to converge as more and more terms are considered in the expansion. Convergence does not occur for steep waves unless a different perturbation parameter from that of Stokes is chosen (Schwartz 1974; Cokelet 1977; Williams 1981, 1985).

(b) The fifth-order Stokes finite-amplitude wave theory is widely used in practical applications both in deep- and shallow-water wave studies. A formulation of Stokes fifth-order theory with good convergence properties has recently been provided (Fenton 1985). Fenton's fifth-order Stokes theory is computationally efficient, and includes closed-form asymptotic expressions for both deep- and shallow-water limits. Kinematics and pressure predictions obtained from this theory compare with laboratory and field measurements better than other nonlinear theories.

(c) In general, the perturbation expansion for velocity potential Φ may be written as

$$\Phi = \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \ldots \qquad \text{(II-1-68)}$$

in which $\varepsilon = ka$ is the *perturbation expansion parameter*. Each term in the series is smaller than the preceding term by a factor of order *ka*. In this expansion, Φ_1 is the first-order theory (linear theory), Φ_2 is the second-order theory, and so on.

(d) Substituting Equation II-1-68 and similar expressions for other wave variables (i.e., surface elevation η, velocities *u* and *w*, pressure *p*, etc.) into the appropriate governing equations and boundary conditions describing the wave motion yields a series of higher-order solutions for ocean waves. Equating the coefficients of equal powers of *ka* gives recurrence relations for each order solution. A characteristic of the perturbation expansion is that each order theory is expressed in terms of the preceding lower order theories (Phillips 1977; Dean and Dalrymple 1991; Mei 1991). The first-order Stokes theory is the linear (Airy) theory.

(e) The Stokes expansion method is formally valid under the conditions that $H/d \ll (kd)^2$ for $kd < 1$ and $H/L \ll 1$ (Peregrine 1972). In terms of the Ursell number U_R these requirements can be met only for $U_R < 79$. This condition restricts the wave heights in shallow water and the Stokes theory is not generally applicable to shallow water. For example, the maximum wave height in shallow water allowed by the second-order Stokes theory is about one-half of the water depth (Fenton 1985). The mathematics of higher-order Stokes theories is cumbersome and is not presented here. See Ippen (1966) for a detailed derivation of the Stokes second-order theory.

(f) In the higher-order Stokes solutions, superharmonic components (i.e., higher frequency components at two, three, four, etc. times the fundamental frequency) arise. These are superposed on the fundamental component predicted by linear theory. Hence, wave crests are steeper and troughs are flatter than the sinusoidal profile (Figure II-1-10). The fifth-order Stokes expansion shows a secondary crest in the wave trough for high-amplitude waves (Peregrine 1972; Fenton 1985). In addition, particle paths for Stokes waves are no longer closed orbits and there is a *drift* or *mass transport* in the direction of wave propagation.

(g) The linear dispersion relation is still valid to second order, and both wavelength and celerity are independent of wave height to this order. At third and higher orders, wave celerity and wavelength depend on wave height, and therefore, for a given wave period, celerity and length are greater for higher waves. Some limitations are imposed on the finite-amplitude Stokes theory in shallow water both by the water depth and amplitude nonlinearities. For steeper waves in shallow water, higher-order terms in Stokes expansion may increase in magnitude to become comparable or larger than the fundamental frequency component (Fenton 1985; Chakrabarti 1987). When this occurs, the Stokes perturbation becomes invalid.

(h) Higher-order Stokes theories include aperiodic (i.e., not periodic) terms in the expressions for water particle displacements. These terms arise from the product of time and a constant depending on the wave period and depth, and give rise to a continuously increasing net particle displacement in the direction of wave propagation. The distance a particle is displaced during one wave period when divided by the wave period gives a mean drift velocity $\bar{U}(z)$, called the *mass transport velocity*. To second-order, the mass transport velocity is

$$\bar{U}(z) = \left(\frac{\pi H}{L}\right)^2 \frac{C}{2} \frac{\cosh\,[4\pi(z + d)/L]}{\sinh^2\,(2\pi d/L)} \tag{II-1-69}$$

indicating that there is a net transport of fluid by waves in the direction of wave propagation. If the mass transport leads to an accumulation of mass in any region, the free surface must rise, thus generating a pressure gradient. A current, formed in response to this pressure gradient, will reestablish the distribution of mass.

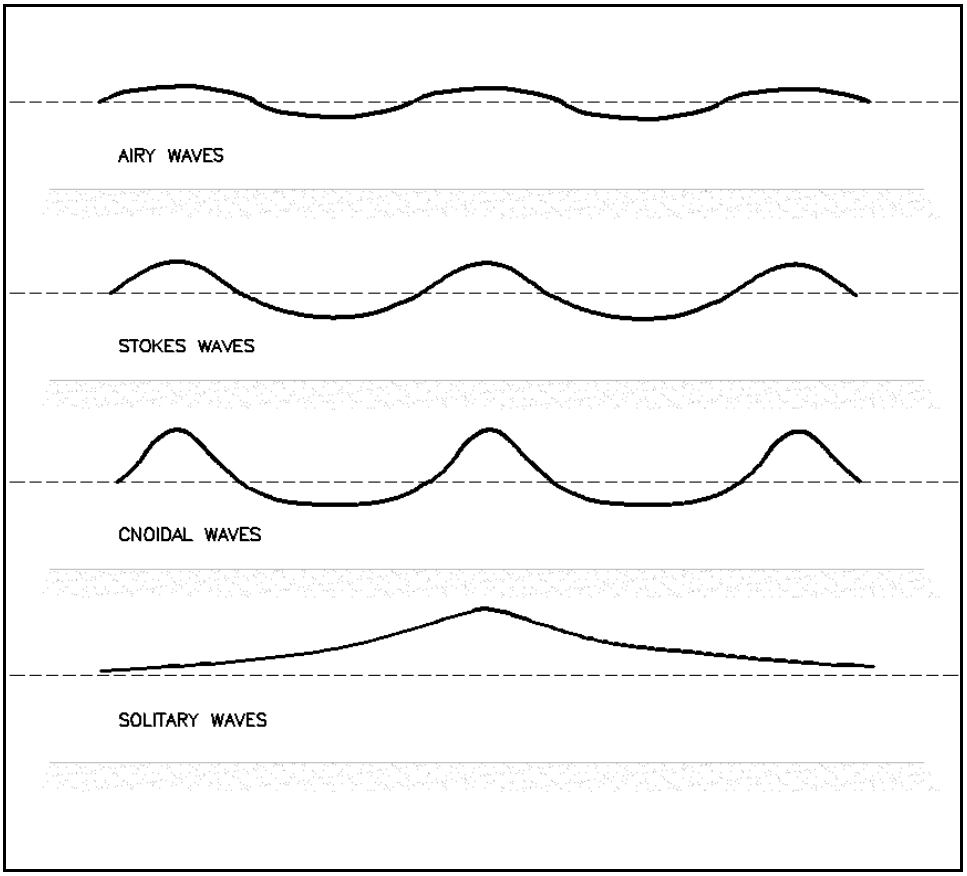

Figure II-1-10. Wave profile shape of different progressive gravity waves

(i) Following Stokes, using higher-order wave theories, both theoretical and experimental studies of mass transport have been conducted (Miche 1944; Ursell 1953; Longuet-Higgins 1953; Russell and Osorio 1958; Isaacson 1978). Results of two-dimensional wave tank experiments where a return flow existed in these studies show that the vertical distribution of the mass transport velocity is modified so that the net transport of water across a vertical plane is zero. For additional information on mass transport, see Dean and Dalrymple (1991).

(3) Subsurface pressure.

(a) Higher-order Stokes theories introduce corrections to the linear wave theory, and often provide more accurate estimates of the wave kinematics and dynamics. For example, the second-order Stokes theory gives the pressure at any distance below the fluid surface as

$$p = \rho g \frac{H}{2} \frac{\cosh[2\pi(z+d)/L]}{\cosh(2\pi d/L)} \cos\theta - \rho g z$$

$$+ \frac{3}{8} \rho g \frac{\pi H^2}{L} \frac{\tanh(2\pi d/L]}{\sinh^2(2\pi d/L} \left(\frac{\cosh[4\pi(z+d)/L]}{\sinh^2(2\pi d/L)} - \frac{1}{3} \right) \cos\ 2\theta \qquad \text{(II-1-70)}$$

$$- \frac{1}{8} \rho g \frac{\pi H^2}{L} \frac{\tanh(2\pi d/L)}{\sinh^2(2\pi d/L)} \left(\cosh \frac{4\pi(z+d)}{L} - 1 \right)$$

(b) The terms proportional to the wave height squared in the above equation represent corrections by the second-order theory to the pressure from the linear wave theory. The third term is the steady component of pressure that corresponds to time-independent terms mentioned earlier.

(c) A direct byproduct of the high-order Stokes expansion is that it provides means for comparing different orders of resulting theories, all of which are approximations. Such comparison is useful to obtain insight about the choice of a theory for a particular problem. Nonetheless, it should be kept in mind that linear (or first-order) theory applies to a wave that is symmetrical about the SWL and has water particles that move in closed orbits. On the other hand, Stokes' higher-order theories predict a wave form that is asymmetrical about the SWL but still symmetrical about a vertical line through the crest and has water particle orbits that are open (Figure II-1-10).

(4) Maximum wave steepness.

(a) A progressive gravity wave is physically limited in height by depth and wavelength. The upper limit or breaking wave height in deep water is a function of the wavelength and, in shallow and transitional water, is a function of both depth and wavelength.

(b) Stokes (1880) predicted theoretically that a wave would remain stable only if the water particle velocity at the crest was less than the wave celerity or phase velocity. If the wave height were to become so large that the water particle velocity at the crest exceeded the wave celerity, the wave would become unstable and break. Stokes found that a wave having a crest angle less than 120 deg would break (angle between two lines tangent to the surface profile at the wave crest). The possibility of the existence of a wave having a crest angle equal to 120 deg is known (Lamb 1945). Michell (1893) found that in deep water the theoretical limit for wave steepness is

$$\left(\frac{H_0}{L_0} \right)_{max} = 0.142 \approx \frac{1}{7} \qquad \text{(II-1-71)}$$

Havelock (1918) confirmed Michell's finding.

(c) Miche (1944) gives the limiting steepness for waves traveling in depths less than $L_0/2$ without a change in form as

$$\left(\frac{H}{L} \right)_{max} = \left(\frac{H_0}{L_0} \right)_{max} \tanh \left(\frac{2\pi d}{L} \right) = 0.142\ \tanh \left(\frac{2\pi d}{L} \right) \qquad \text{(II-1-72)}$$

Laboratory measurements indicate that Equation II-1-72 is in agreement with an envelope curve to laboratory observations (Dean and Dalrymple 1991).

e. Other wave theories.

(1) Introduction.

(a) Extension of the Stokes theory to higher orders has become common with computers, but the mathematics involved is still tedious. Variations of the Stokes theory have been developed in the last three decades oriented toward computer implementation. For example, Dean (1965) used the stream function in place of the velocity potential to develop the stream function theory. Dean (1974) did a limited comparison of measured horizontal particle velocity in a wave tank with the tenth-order stream function theory and several other theories. Forty cases were tabulated in dimensionless form to facilitate application of this theory.

(b) Others (Dalrymple 1974a; Chaplin 1980; Reinecker and Fenton 1981) developed variations of the stream function theory using different numerical methods. Their studies included currents. For near-breaking waves, Cokelet (1977) extended the method of Schwartz (1974) for steep waves for the full range of water depth and wave heights. Using a 110th-order theory for waves up to breaking, Cokelet successfully computed the wave profile, wave celerity, and various integral properties of waves, including the mean momentum, momentum flux, kinetic and potential energy, and radiation stress.

(2) Nonlinear shallow-water wave theories.

(a) Stokes' finite amplitude wave theory is applicable when the depth to wavelength ratio d/L is greater than about 1/8 or $kd > 0.78$ or $U_r < 79$. For longer waves a different theory must be used (Peregrine 1976). As waves move into shallow water, portions of the wave travel faster because of amplitude dispersion or waves travel faster because they are in deeper water. Waves also feel the effects of frequency dispersion less in shallow water, e.g., their speed is less and less influenced by water depth.

(b) For the mathematical representation of waves in shallow water, a different perturbation parameter should be used to account for the combined influence of amplitude and frequency dispersion (Whitham 1974; Miles 1981; Mei 1991). This can be achieved by constructing two perturbation parameters whose ratio is equivalent to the Ursell parameters (Peregrine 1972). The set of equations obtained in this manner are termed the *nonlinear shallow-water wave equations*. Some common wave theories based on these equations are briefly described in the following sections.

(3) Korteweg and de Vries and Boussinesq wave theories.

(a) Various shallow-water equations can be derived by assuming the pressure to be hydrostatic so that vertical water particle accelerations are small and imposing a horizontal velocity on the flow to make it steady with respect to the moving reference frame. The horizontal velocity might be the velocity at the SWL, at the bottom, or the velocity averaged over the depth. If equations are written in terms of depth-averaged velocity $\overline{u}$ they become:

$$\begin{gathered}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(d + \eta)\overline{u} = 0 \\ \frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + g\frac{\partial \eta}{\partial x} = \frac{1}{3}d^2\frac{\partial^3 \overline{u}}{\partial x^2 \partial t}\end{gathered} \tag{II-1-73}$$

which are termed the *Boussinesq equations* (Whitham 1967; Peregrine 1972; Mei 1991). Originally, Boussinesq used the horizontal velocity at the bottom. Eliminating ū yields (Miles 1979, 1980, 1981)

$$\frac{\partial^2 \eta}{\partial t^2} - gd\frac{\partial^2 \eta}{\partial x^2} = gd\frac{\partial^2}{\partial x^2}\left(\frac{3}{2}\frac{\eta^2}{d} + \frac{1}{3}d^2\frac{\partial^2 \eta}{\partial x^2}\right) \qquad \text{(II-1-74)}$$

A periodic solution to Equation II-1-74 is of the form

$$\begin{aligned} \eta &= a\ e^{\,i(kx - \omega t)} = a\ \cos\ \theta \\ \overline{u} &= U_0\ e^{\,i(kx - \omega t)} = U_0\ \cos\ \theta \end{aligned} \qquad \text{(II-1-75)}$$

which has a dispersion relation and an approximation to it given by

$$C = \frac{C_s}{\left[1 + \frac{1}{3}(kd)^2\right]^{1/2}} \approx C_s\left[1 - \frac{1}{3}(kd)^2 + ...\right] \qquad \text{(II-1-76)}$$

The term $1/3\ (kd)^2$ in Equation II-1-76 represents the dispersion of wave motion.

(c) The most elementary solution of the Boussinesq equation is the *solitary wave* (Russell 1844; Fenton 1972; Miles 1980). A solitary wave is a wave with only crest and a surface profile lying entirely above the SWL. Fenton's solution gives the maximum solitary wave height, $H_{max} = 0.85$ d and maximum propagation speed $C^2_{max} = 1.7$ gd. Earlier research studies using the solitary waves obtained $H_{max} = 0.78$ d and $C^2_{max} = 1.56$ gd. The maximum solitary-amplitude wave is frequently used to calculate the height of breaking waves in shallow water. However, subsequent research has shown that the highest solitary wave is not necessarily the most energetic (Longuet-Higgins and Fenton 1974).

(4) Cnoidal wave theory.

(a) Korteweg and de Vries (1895) developed a wave theory termed the *cnoidal theory*. The cnoidal theory is applicable to finite-amplitude shallow-water waves and includes both nonlinearity and dispersion effects. Cnoidal theory is based on the Boussinesq, but is restricted to waves progressing in only one direction. The theory is defined in terms of the *Jacobian elliptic function*, *cn*, hence the name cnoidal. Cnoidal waves are periodic with sharp crests separated by wide flat troughs (Figure II-1-10).

(b) The approximate range of validity of the cnoidal theory is $d/L < 1/8$ when the Ursell number $U_R > 20$. As wavelength becomes long and approaches infinity, cnoidal wave theory reduces to the solitary wave theory, which is described in the next section. Also, as the ratio of wave height to water depth becomes small (infinitesimal wave height), the wave profile approaches the sinusoidal profile predicted by the linear theory.

(c) Cnoidal waves have been studied extensively by many investigators (Keulegan and Patterson 1940; Keller 1948; Laitone 1962) who developed first- through third-order approximations to the cnoidal wave theory. Wiegel (1960) summarized the principal results in a more usable form by presenting such wave characteristics as length, celerity, and period in tabular and graphical form to facilitate application of cnoidal theory.

(d) Wiegel (1964) further simplified the earlier works for engineering applications. Recent additional improvements to the theory have been made (Miles 1981; Fenton 1972, 1979). Using a Rayleigh-Boussinesq series, Fenton (1979) developed a generalized recursion relationship for the KdV solution of any order. Fenton's fifth- and ninth-order approximations are frequently used in practice. A summary of formulas of the cnoidal wave theory are provided below. See Fenton (1979), Fenton and McKee 1990), and Miles (1981) for a more comprehensive theoretical presentation.

(e) Long, finite-amplitude waves of permanent form propagating in shallow water may be described by cnoidal wave theory. The existence in shallow water of such long waves of permanent form may have first been recognized by Boussinesq (1871). However, the theory was originally developed by Korteweg and de Vries (1895).

(f) Because local particle velocities, local particle accelerations, wave energy, and wave power for cnoidal waves are difficult to describe such descriptions are not included here, but can be obtained in graphical form from Wiegel (1960, 1964). Wave characteristics are described in parametric form in terms of the modules k of the *elliptic integrals*. While k itself has no physical significance, it is used to express the relationships between various wave parameters. Tabular presentations of the elliptic integrals and other important functions can be obtained from the above references. The ordinate of the water surface y_s measured above the bottom is given by

$$y_s = y_t + H\, cn^2 \left[2K(k) \left(\frac{x}{L} - \frac{t}{T} \right) \;,\; k \right] \qquad \text{(II-1-77)}$$

where

y_t = distance from the bottom to the wave trough

H = trough to crest wave height

cn = elliptic cosine function

$K(k)$ = complete elliptic integral of the first kind

k = modulus of the elliptic integrals

(g) The argument of cn^2 is frequently denoted simply by ( ); thus, Equation II-1-77 above can be written as

$$y_s = y_t + H\, cn^2(\quad) \qquad \text{(II-1-78)}$$

(h) The elliptic cosine is a periodic function where cn^2 [2K(k) ((x/L) - (t/T)] has a maximum amplitude equal to unity. The modulus k is defined over the range 0 and 1. When $k = 0$, the wave profile becomes a sinusoid, as in the linear theory; when $k = 1$, the wave profile becomes that of a solitary wave.

(i) The distance from the bottom to the wave trough y_t, as used in Equations II-1-77 and II-1-78, is given by

$$\frac{y_t}{d} = \frac{y_c}{d} - \frac{H}{d} = \frac{16d^2}{3L^2} K(k)\; [K(k) - E(k)] + 1 - \frac{H}{d} \qquad \text{(II-1-79)}$$

where y_c is the distance from the bottom to the crest, and *E(k)* the complete elliptic integral of the second kind. Wavelength is given by

$$L = \sqrt{\frac{16d^3}{3H}}\ k\ K(k) \tag{II-1-80}$$

and wave period by

$$T\sqrt{\frac{g}{d}} = \sqrt{\frac{16y_t}{3H}}\ \frac{d}{y_t}\left[\frac{k\ K(k)}{1 + \dfrac{H}{y_t\ k^2}\left(\dfrac{1}{2} - \dfrac{E(k)}{K(k)}\right)}\right] \tag{II-1-81}$$

Note that cnoidal waves are periodic and of permanent form; thus L = CT (see Figure II-1-10).

(j) Pressure under a cnoidal wave at any elevation y above the bottom depends on the local fluid velocity, and is therefore complex. However, it may be approximated in a hydrostatic form as

$$p = \rho g\ (y_s - y) \tag{II-1-82}$$

i.e., the pressure distribution may be assumed to vary linearly from $\rho g y_s$ at the bed to zero at the surface.

(k) Wave profiles obtained from different wave theories are sketched in Figure II-1-10 for comparison. The linear profile is symmetric about the SWL. The Stokes wave has higher more peaked crests and shorter, flatter troughs. The cnoidal wave crests are higher above the SWL than the troughs are below the SWL. Cnoidal troughs are longer and flatter and crests are sharper and steeper than Stokes waves. The solitary wave, a form of the cnoidal wave described in the next section, has all of its profile above the SWL.

(l) Figures II-1-11 and II-1-12 show the dimensionless cnoidal wave surface profiles for various values of the square of the modulus of the elliptic integrals k^2, while Figures II-1-13 to II-1-16 present dimensionless plots of the parameters which characterize cnoidal waves. The ordinates of Figures II-1-13 and II-1-14 should be read with care, since values of k^2 are extremely close to 1.0 ($k^2 = 1 - 10^{-1} = 1 - 0.1 = 0.90$). It is the exponent α of $k^2 = 1 - 10^{-\alpha}$ that varies along the vertical axis of Figures II-1-13 and II-1-14.

(m) Ideally, shoaling computations might be performed using a higher-order cnoidal wave theory since this theory is able to describe wave motion in relatively shallow water. Simple, completely satisfactory procedures for applying cnoidal wave theory are not available. Although linear wave theory is often used, cnoidal theory may be applied for practical situations using Figures such as II-1-11 to II-1-16. The following problem illustrates the use of these figures.

(n) There are two limits to the cnoidal wave theory. The first occurs when the period of the function cn is infinite when $k = 1$. This corresponds to a solitary wave. As the wavelength becomes infinite, the cnoidal theory approaches the solitary wave theory. The second limit occurs for $k = 0$ where the cnoidal wave approaches the sinusoidal wave. This happens when the wave height is small compared to water depth and the cnoidal theory reduces to the linear theory.

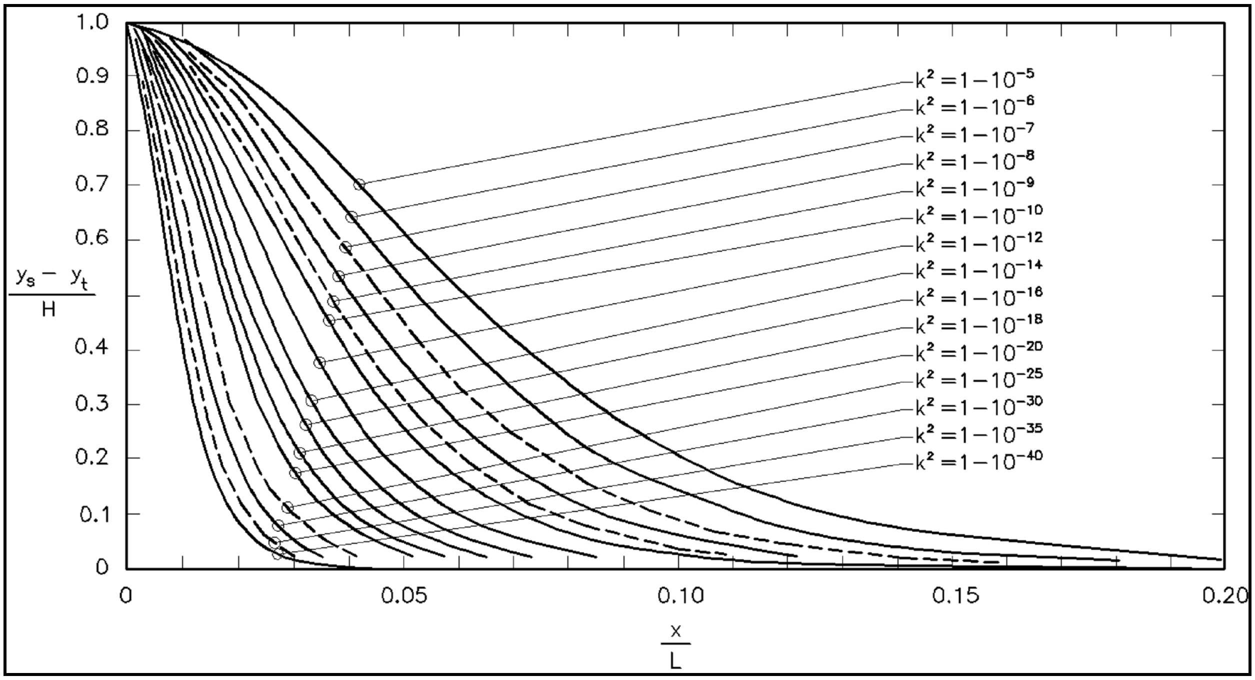

Figure II-1-11. Normalized surface profile of the cnoidal wave (Wiegel 1960). For definition of variables see Part II-1-2.e.(3)

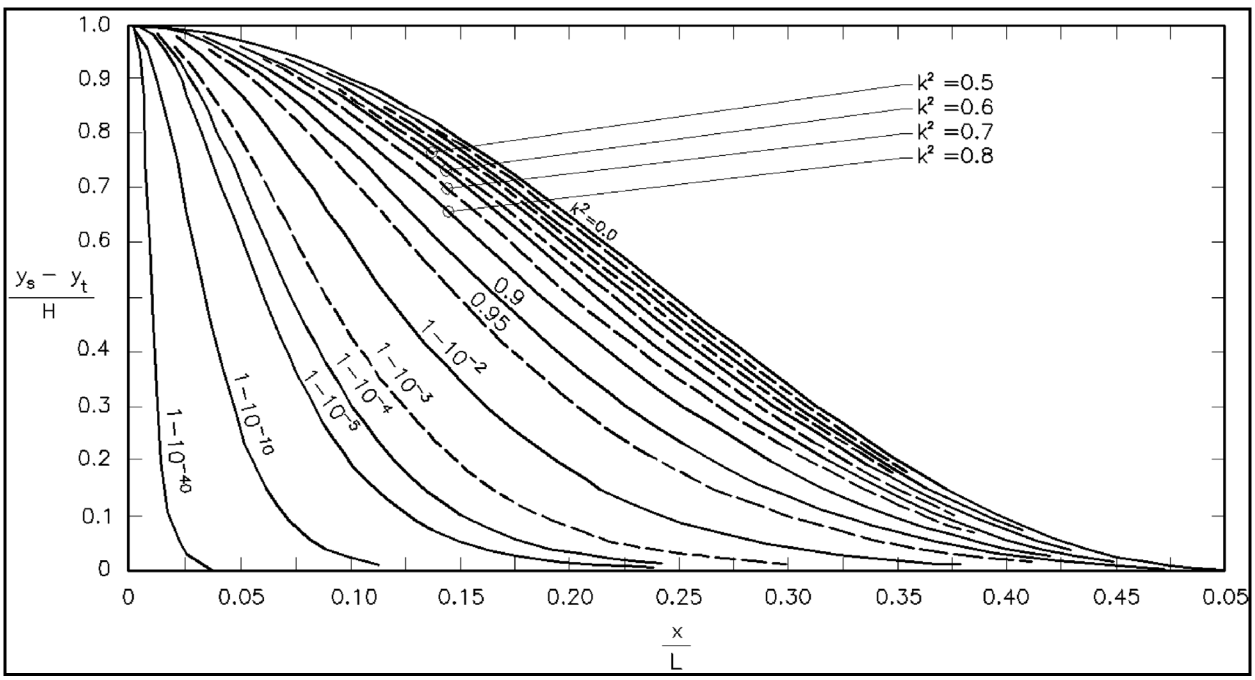

Figure II-1-12. Normalized surface profile of the cnoidal wave for higher values of k^2 and X/L (Wiegel 1960)

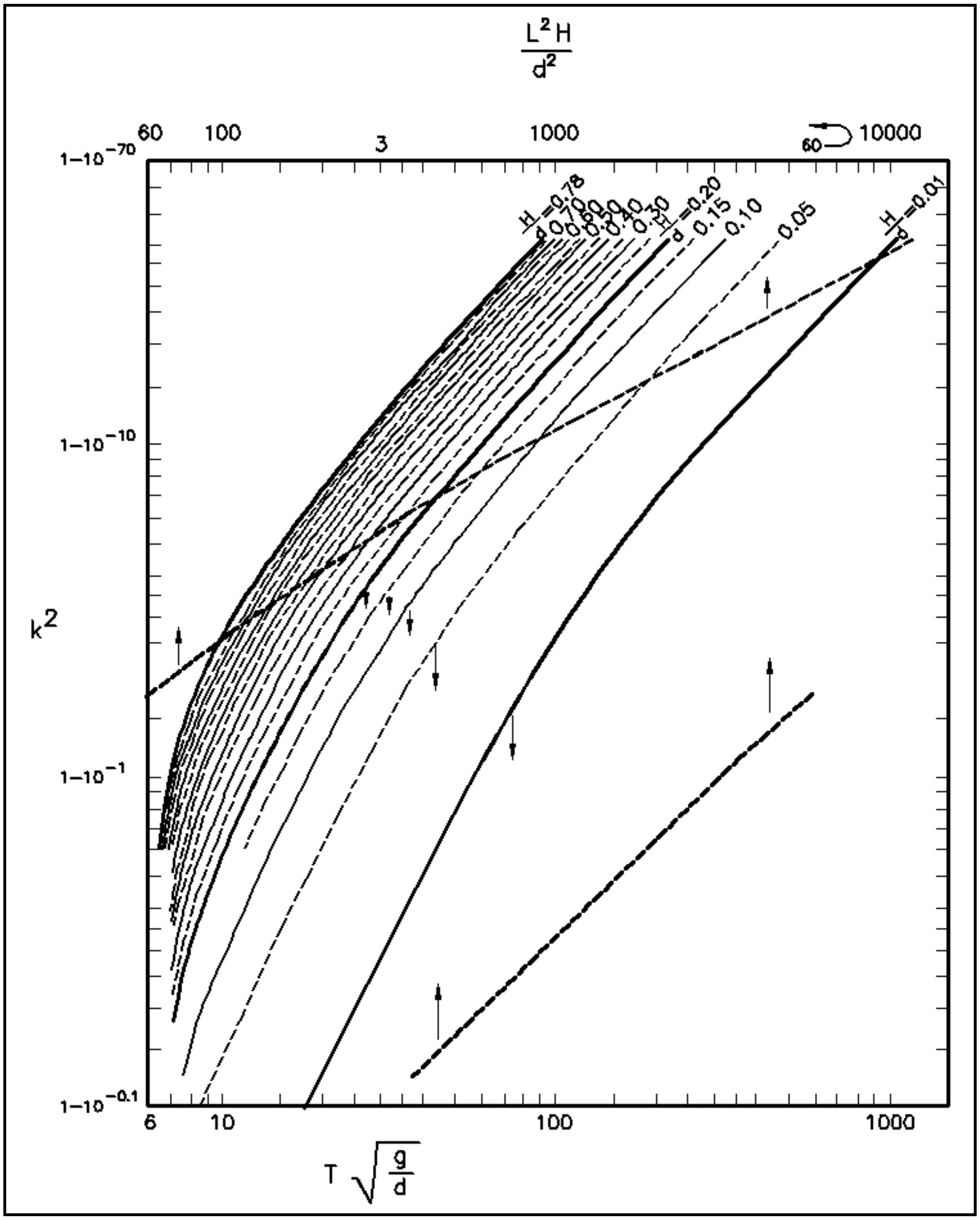

Figure II-1-13. k^2 versus L^2H/d^3, and k^2 versus $T\sqrt{g/d}$ and H/d (Wiegel 1960)

(5) Solitary wave theory.

(a) Waves considered in the previous sections were oscillatory or nearly oscillatory waves. The water particles move backward and forward with the passage of each wave, and a distinct wave crest and wave trough are evident. A solitary wave is neither oscillatory nor does it exhibit a trough. In the pure sense, the solitary wave form lies entirely above the still-water level. The solitary wave is a wave of translation because the water particles are displaced a distance in the direction of wave propagation as the wave passes.

(b) The *solitary wave* was discovered by Russell (1844). Boussinesq (1871), Rayleigh (1876), Keller (1948), and Munk (1949) performed pioneering theoretical studies of solitary waves. More recent analyses

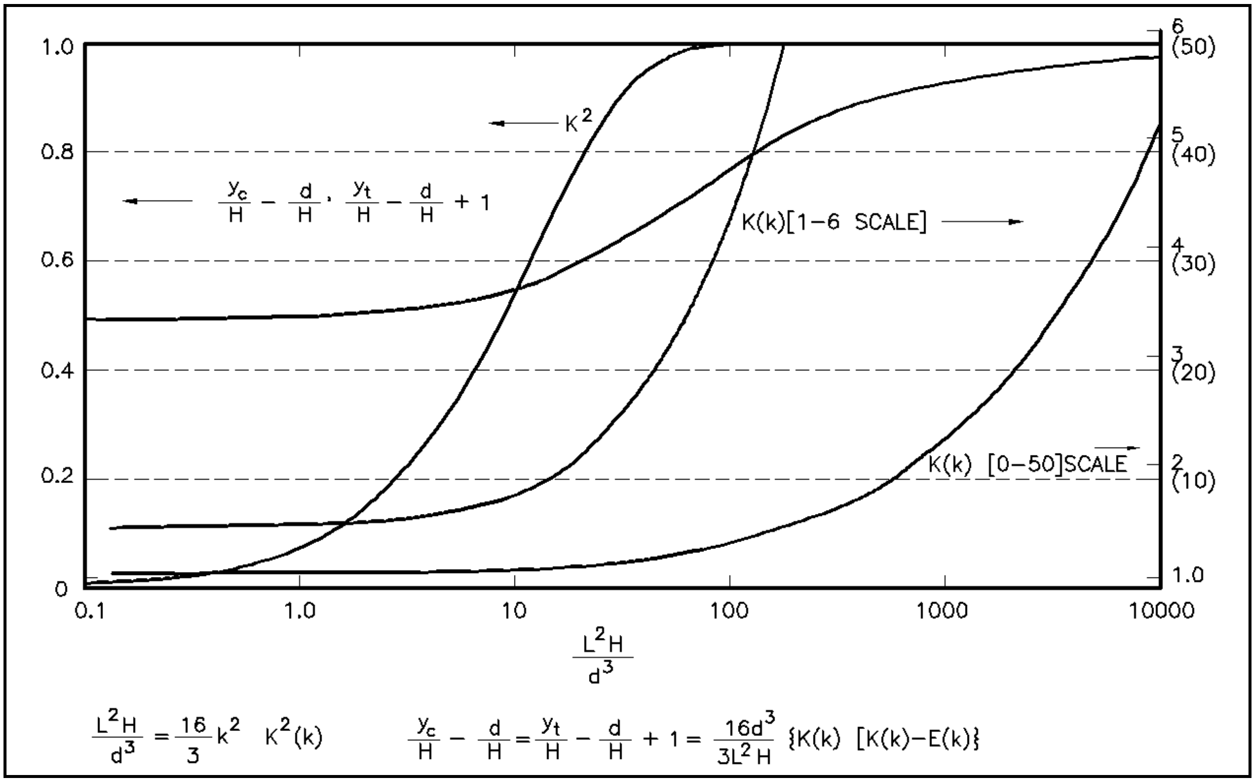

Figure II-1-14. Relationship among L^2H/d^3 and the square of the elliptic modulus (k^2), y_c/H, y_t/H, and K(k) (Wiegel 1960)

of solitary waves were performed by Fenton (1972), Longuet-Higgins and Fenton (1974), and Byatt-Smith and Longuet-Higgins (1976). The first systematic observations and experiments on solitary waves can probably be attributed to Russell (1838, 1844), who first recognized the existence of a solitary wave.

(c) In nature it is difficult to form a truly solitary wave, because at the trailing edge of the wave there are usually small dispersive waves. However, long waves such as tsunamis and waves resulting from large displacements of water caused by such phenomena as landslides and earthquakes sometimes behave approximately like solitary waves. When an oscillatory wave moves into shallow water, it may often be approximated by a solitary wave (Munk 1949). As an oscillatory wave moves into shoaling water, the wave amplitude becomes progressively higher, the crests become shorter and more pointed, and the trough becomes longer and flatter.

(d) Because both wavelength and period of solitary waves are infinite, only one parameter H/d is needed to specify a wave. To lowest order, the solitary wave profile varies as $sech^2 q$ (Wiegel 1964), where $q = (3H/d)^{1/2}\,(x-Ct)/2d$ and the free-surface elevation, particle velocities, and pressure may be expressed as

$$\frac{\eta}{H} = \frac{u}{\sqrt{gd}\ \dfrac{H}{d}} \qquad \text{(II-1-83)}$$

$$\frac{u}{\sqrt{gd}}\ \frac{H}{d} = \frac{\Delta p}{\rho g H} \qquad \text{(II-1-84)}$$

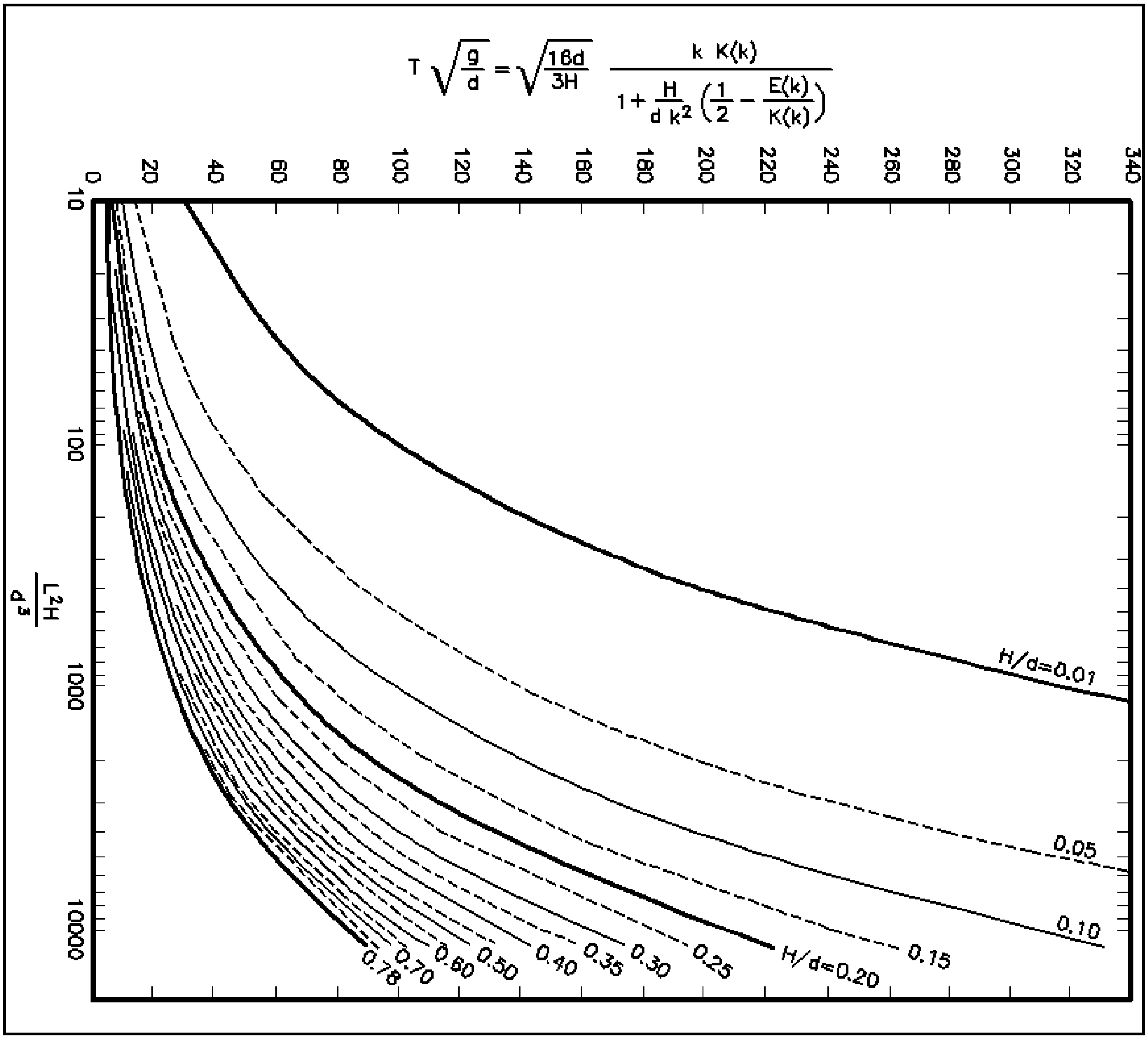

Figure II-1-15. Relationships among $T\sqrt{g/d}$, L^2H/d^3, and H/d (Wiegel 1960)

$$\frac{\Delta p}{\rho g H} = sech^2\, q \qquad \text{(II-1-85)}$$

where Δp is the difference in pressure at a point due to the presence of the solitary wave.

(e) To second approximation (Fenton 1972), this difference is given by

$$\frac{\Delta p}{\rho g H} = 1 - \frac{3}{4}\frac{H}{d}\left[1 - \left(\frac{Y_s}{d}\right)^2\right] \qquad \text{(II-1-86)}$$

where y_s = the height of the surface profile above the bottom. The wave height H required to produce Δp on the seabed can be estimated from

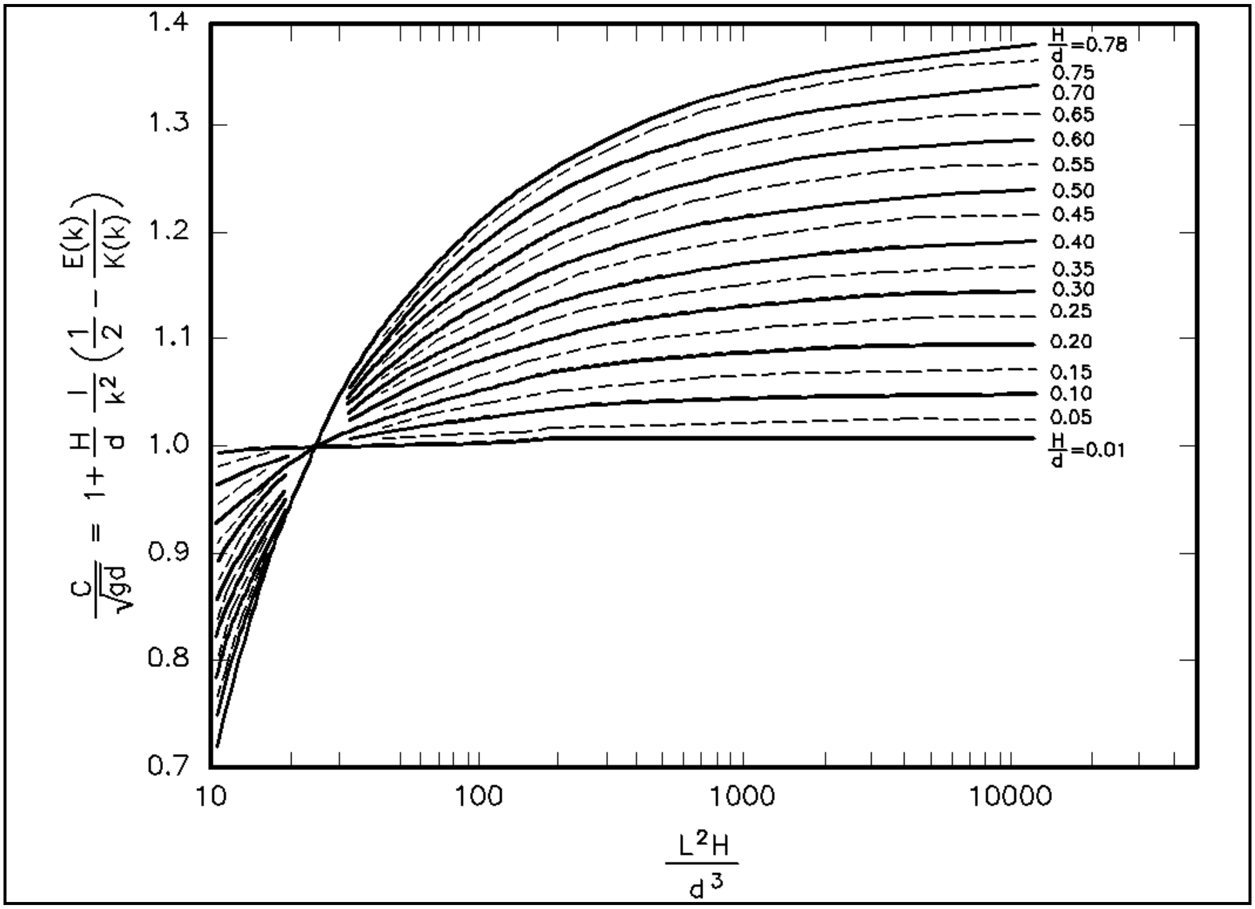

Figure II-1-16. Relationship between cnoidal wave velocity and L^2H/d^3 (Wiegel 1960)

$$\frac{\Delta p}{\rho g H} = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{3\Delta p}{\rho g d}} \qquad \text{(II-1-87)}$$

(f) Since the solitary wave has horizontal particle velocities only in the direction of wave advance, there is a net displacement of fluid in the direction of wave propagation.

(g) The solitary wave is a limiting case of the cnoidal wave. When $k^2 = 1$, $K(k) = K(1) = \infty$, and the elliptic cosine reduces to the hyperbolic secant function and the water surface y_s measured above the bottom reduces to

$$y_s = d + H\, sech^2\left[\sqrt{\frac{3}{4}\frac{H}{d^3}}\,(x - Ct)\right] \qquad \text{(II-1-88)}$$

(h) The free surface is given by

$$\eta = H\, sech^2\left[\sqrt{\frac{3}{4}\frac{H}{d^3}}\,(x - Ct)\right] \qquad \text{(II-1-89)}$$

EXAMPLE PROBLEM II-1-5

FIND:

(a) Using cnoidal wave theory, find the wavelength L and compare this length with the length determined using Airy theory.

(b) Determine the celerity C. Compare this celerity with the celerity determined using Airy theory.

(c) Determine the distance above the bottom of the wave crest y_c and wave trough y_t .

(d) Determine the wave profile.

GIVEN:

A wave traveling in water depth d = 3 m (9.84 ft), with a period T = 15 sec, and a height H = 1.0 m (3.3 ft).

SOLUTION:

(a) Calculate

$$\frac{H}{d} = \frac{1}{3} = 0.33$$

and

$$T \sqrt{\frac{g}{d}} = 15 \sqrt{\frac{9.8}{3}} = 27.11$$

From Figure II-1-13, enter *H/d* and *T* to determine the square of the modulus of the complete elliptical integrals, k^2:

$$k^2 = 1 - 10^{-5}$$

Entering both Figures II-1-13 and II-1-14 with the value of k^2 gives

$$\frac{L^2 H}{d^3} = 290$$

or

$$L = \sqrt{\frac{290\ d^3}{H}} = \sqrt{\frac{290\ (3)^3}{1}}$$

Example Problem II-1-5 (Continued)

Example Problem II-1-5 (Continued)

which gives L = 88.5 m (290.3 ft). The wavelength from the linear (Airy) theory is

$$L = \frac{gT^2}{2\pi} \tanh\left(\frac{2\pi d}{L}\right) = 80.6\ m\ (264.5\ ft)$$

To check whether the wave conditions are in the range for which cnoidal wave theory is valid, calculate *d/L* and the *Ursell number* = L^2H/d^3:

$$\frac{d}{L} = \frac{3}{88.5} = 0.0339 < \frac{1}{8} \qquad \text{0.K.}$$

$$\frac{L^2H}{d^3} = \frac{1}{\left(\frac{d}{L}\right)^2}\left(\frac{H}{d}\right) = 290 > 26 \qquad \text{0.K.}$$

Therefore, cnoidal theory is applicable.

(b) Wave celerity is given by

$$C = \frac{L}{T} = \frac{88.5}{15} = 5.90\ m/s\ (19.36\ ft/s)$$

while the linear theory predicts

$$C = \frac{L}{T} = \frac{80.6}{15} = 5.37\ m/s\ (17.63\ ft/s)$$

Thus, if it is assumed that the wave period is the same for cnoidal and Airy theories, then

$$\frac{C_{cnoidal}}{C_{Airy}} = \frac{L_{cnoidal}}{L_{Airy}} \approx 1$$

(c) The percentage of the wave height above the SWL may be determined from Figure II-1-11 or II-1-12. Entering these figures with L^2H/D^3 = 290, the value of $(y_c - d)/H$ is found to be 0.865, or 86.5 percent. Therefore,

$$y_c = 0.865\ H + d$$

$$y_c = 0.865(1) + 3 = 0.865 + 3 = 3.865\ m\ (12.68\ ft)$$

Example Problem II-1-5 (Continued)

Example Problem II-1-5 (Concluded)

Also from Figure II-1-11 or II-1-12,

$$\frac{(y_t - d)}{H} + 1 = 0.865$$

thus,

$$y_t = (0.865 - 1)(1) + 3 = 2.865 \ m \ (9.40 \ ft)$$

(d) The dimensionless wave profile is given in Figures II-1-11 and II-1-12 for $k^2 = 1 - 10^{-5}$. The results obtained in (c) above can also be checked by using Figures II-1-11 and II-1-12. For the wave profile obtained with $k^2 = 1 - 10^{-5}$, the SWL is approximately 0.14H above the wave trough or 0.86H below the wave crest.

The results for the wave celerity determined under (b) above can now be checked with the aid of Figure II-1-16. Calculate

$$\frac{H}{y_t} = \frac{(1)}{2.865} = 0.349$$

Entering Figure II-1-16 with

$$\frac{L^2 H}{d^3} = \frac{(1)}{2.865} = 0.349$$

and

$$\frac{H}{y_t} = 0.349$$

it is found that

$$\frac{C}{\sqrt{g \ y_t}} = 1.126$$

Therefore,

$$C = 1.126 \sqrt{(9.8)(2.865)} = 5.97 \ m/s \ (19.57 \ ft/s)$$

The differences between this number and the 5.90 m /sec (18.38 ft/s) calculated under (b) above is the result of small errors in reading the curves.

where the origin of x is at the wave crest. The volume of water within the wave above the still-water level per unit crest width is

$$V = \left[\frac{16}{3}\ d^3\ H\right]^{\frac{1}{2}} \tag{II-1-90}$$

(i) An equal amount of water per unit crest length is transported forward past a vertical plane that is perpendicular to the direction of wave advance. Several relations have been presented to determine the celerity of a solitary wave; these equations differ depending on the degree of approximation. Laboratory measurements suggest that the simple expression

$$C = \sqrt{g(H + d)} \tag{II-1-91}$$

gives a reasonably accurate approximation to the celerity of solitary wave.

(j) The water particle velocities for a solitary wave (Munk 1949), are

$$u = CN\ \frac{1 + \cos(My/d)\ \cosh(Mx/d)}{[\cos(My/d) + \cosh(Mx/D)]^2} \tag{II-1-92}$$

$$w = CN\ \frac{\sin(My/d)\ \sinh(Mx/d)}{[\cos(My/d) + \cosh(Mx/D)]^2} \tag{II-1-93}$$

where M and N are the functions of H/d shown in Figure II-1-17, and y is measured from the bottom. The expression for horizontal velocity u is often used to predict wave forces on marine structures situated in shallow water. The maximum velocity u_{max} occurs when x and t are both equal to zero; hence,

$$u_{\max} = \frac{CN}{1 + \cos(My/d)} \tag{II-1-94}$$

(h) Total energy in a solitary wave is about evenly divided between kinetic and potential energy. Total wave energy per unit crest width is

$$E = \frac{8}{3\sqrt{3}}\ \rho g H^{\frac{3}{2}}\ d^{\frac{3}{2}} \tag{II-1-95}$$

and the pressure beneath a solitary wave depends on the local fluid velocity, as does the pressure under a cnoidal wave; however, it may be approximated by

$$p = \rho g\ (y_s - y) \tag{II-1-96}$$

(l) Equation II-1-96 is identical to that used to approximate the pressure beneath a cnoidal wave.

(m) As a solitary wave moves into shoaling water it eventually becomes unstable and breaks. A solitary wave breaks when the water particle velocity at the wave crest becomes equal to the wave celerity. This occurs when (Miles 1980, 1981)

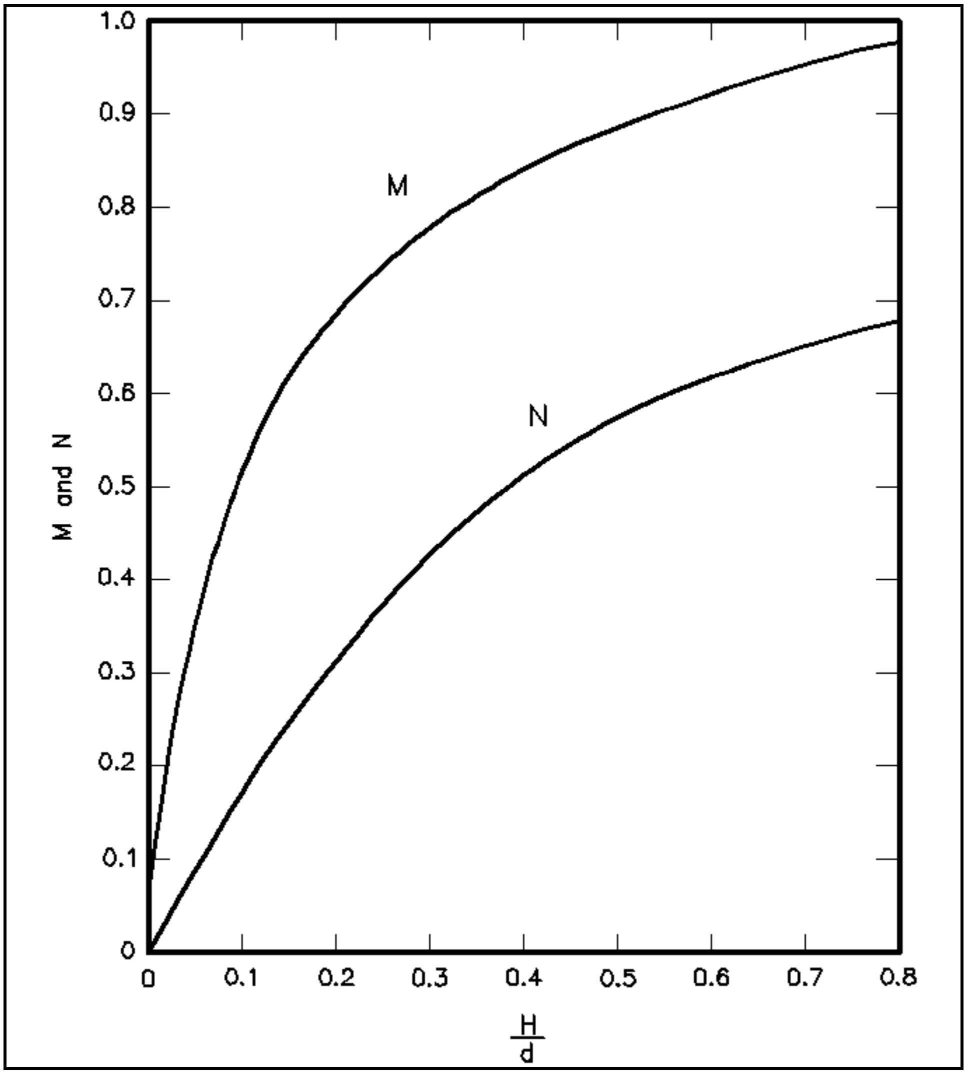

Figure II-1-17. Functions M and N in solitary wave theory (Munk 1949)

$$\left(\frac{H}{d}\right)_{\max} = 0.78 \tag{II-1-97}$$

(n) Laboratory studies have shown that the value of $(H/d)_{max}$ = 0.78 agrees better with observations for oscillatory waves than for solitary waves and that the nearshore slope has a substantial effect on this ratio. Other factors such as bottom roughness may also be involved. Tests of periodic waves with periods from 1 to 6 sec on slopes of m = 0.0, 0.05, 0.10, and 0.20 have shown (SPM 1984) that H_b/d_b ratios are approximately equal to 0.83, 1.05, 1.19, and 1.32, respectively. Tests of single solitary waves on slopes from m = 0.01 to m = 0.20 (SPM 1984) indicate an empirical relationship between the slope and the breaker height-to-water depth ratio given by

$$\frac{H_b}{d_b} = 0.75 + 25m - 112m^2 + 3870m^3 \tag{II-1-98}$$

in which waves did not break when the slope *m* was greater than about 0.18 and that as the slope increased the breaking position moved closer to the shoreline. This accounts for the large values of H_b/d_b for large slopes; i.e., as $d_b \rightarrow 0$. For some conditions, Equations II-1-97 and II-1-98 are unsatisfactory for predicting breaking depth. Further discussion of the breaking of waves with experimental results is provided in Part II-4.

(6) Stream-function wave theory. Numerical approximations to solutions of hydrodynamic equations describing wave motion have been proposed and developed. Some common theories and associated equations are listed in Table II-1-2. The approach by Dean (1965, 1974), termed a *symmetric, stream-function theory*, is a nonlinear wave theory that is similar to higher order Stokes' theories. Both are constructed of sums of *sine* or *cosine* functions that satisfy the original differential equation (Laplace equation). The theory, however, determines the coefficient of each higher order term so that a best fit, in the least squares sense, is obtained to the theoretically posed, dynamic, free-surface boundary condition. Assumptions made in the theory are identical to those made in the development of the higher order Stokes' solutions. Consequently, some of the same limitations are inherent in the stream-function theory, and it represents an alternative solution to the equations used to approximate the wave phenomena. However, the stream-function representation had successfully predicted the wave phenomena observed in some laboratory wave studies (Dean and Dalrymple 1991), and thus it may possibly describe naturally occurring wave phenomena.

Table II-1-2
Boundary Value Problem of Water Wave Theories (Dean 1968)

		Exactly Satisfies			
Theory		**DE**	**BBC**	**KFSBC**	**DFSBC**
Linear wave theory		X	X	-	-
Third-order Stokes		X	X	-	-
Fifth-order Stokes		X	X	-	-
First-order cnoidal		-	X	-	-
Second-order cnoidal		-	X	-	-
Stream function theory	numerical wave	X	X	X	-

DE = Differential equation.
BBC = Bottom boundary condition.
KFSBC = Kinematic free surface boundary condition.
DFSBC = Dynamic free surface boundary condition.
X = Exactly satisfies.

(7) Fourier approximation -- Fenton's theory.

(a) Fenton's Fourier series theory, another theory developed in recent years (Fenton 1988), is somewhat similar to Dean's stream function theory, but it appears to describe oceanic waves at all water depths better than all previous similar theories.

(b) The long, tedious computations involved in evaluating the terms of the series expansions that make up the higher order stream-function theory of Dean had in the past limited its use to either tabular or graphical presentations of the solutions. These tables, their use, and their range of validity may be found elsewhere (Dean 1974).

(c) Stokes and cnoidal wave theories yield good approximations for waves over a wide range of depths if high-order expansions are employed. Engineering practice has relied on the Stokes fifth-order theory (Skjelbreia and Hendrickson 1961), and the stream function theory (Dean 1974). These theories are applicable to deepwater applications. An accurate steady wave theory may be developed by numerically solving the full nonlinear equations with results that are applicable for short waves (deep water) and for long waves (shallow water). This is the Fourier approximation method. The method is termed *Fenton's theory* here. Any periodic function can be approximated by Fourier series, provided the coefficients of the series can be found. In principal, the coefficients are found numerically. Using this approach, Chappelear (1961) developed a Fourier series solution by adopting the velocity potential as the primary field variable. Dean (1965, 1974) developed the stream function theory. The solutions by both Chappelear and Dean successively correct an initial estimate to minimize errors in the nonlinear free-surface boundary conditions.

(d) A simple Fourier approximation wave theory was introduced by Rienecker and Fenton (1981) and was subsequently improved by Fenton (1985, 1988; Fenton and McKee 1990). It is an improved numerical theory that has a range of applicability broader than the Stokes and cnoidal theories. Details of the theory are given by Reinecker and Fenton (1981) and Fenton (1985, 1988; Fenton and McKee 1990). Sobey et al. (1987) recasted Fenton's work into a standardized format including currents in the formulation up to fifth order. The theory has been implemented to calculate wave kinematics and the loading of offshore structures (Demirbilek 1985). For coastal applications, a PC-based computer code of Fenton's theory is available in the Automated Coastal Engineering System (ACES) (Leenknecht, Szuwalski, and Sherlock 1992). A brief description of Fenton's theory is given here; details are provided in ACES.

(e) Fenton's Fourier approximation wave theory satisfies field equations and boundary conditions to a specified level of accuracy. The hydrodynamic equations governing the problem are identical to those used in Stokes' theory (Table II-1-2). Various approximations introduced in earlier developments are indicated in the table. Like other theories, Fenton's theory adopts the same field equation and boundary conditions. There are three major differences between Fenton's theory and the others. First, Fenton's theory is valid for deep- and shallow-water depths, and any of the two quantities' wave height, period or energy flux can be specified to obtain a solution. Second, the Fourier coefficients are computed numerically with efficient algorithms. Third, the expansion parameter for the Fourier coefficients is $\epsilon = kH/2$ rather than $\epsilon = ka$, which is used in Stokes theories. The coefficients are found numerically from simultaneous algebraic equations by satisfying two nonlinear free-surface boundary conditions and the dispersion relationship. Finding the coefficients requires that wave height, wave period, water depth, and either the Eulerian current or the depth-averaged mass transport velocity be specified.

(f) In Fenton's theory, the governing field equation describing wave motion is the two-dimensional (x,z in the Cartesian frame) Laplace's equation, which in essence is an expression of the conservation of mass:

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial z^2} = 0 \qquad \text{(II-1-99)}$$

where Ψ is the stream function. Ψ is a periodic function that describes wave motion in space and time, which also relates to the flow rate.

(g) Wave motion is a boundary-value problem, and its solution requires specifying realistic boundary conditions. These boundary conditions are usually imposed at the free surface and sea bottom. Since the seabed is often impermeable, flow rate through the sea bottom must be zero. Therefore, the bottom boundary condition may be stated in terms of Ψ as

$$\Psi(x,-d) = 0 \quad \text{at } z = -d \qquad \text{(II-1-100)}$$

(h) Two boundary conditions, *kinematic* and *dynamic*, are needed at the free surface. The kinematic condition states that water particles on the free surface remain there, and consequently, flow rate through the surface boundary must be zero. The net flow Q between the sea surface and seabed may be specified as

$$\Psi(x,\eta) = -Q \qquad \text{at } z = \eta \tag{II-1-101}$$

where η is the sea surface elevation. The dynamic free-surface boundary condition is an expression of specifying the pressure at the free surface that is constant and equal to the atmospheric pressure. In terms of the stream function Ψ this condition may be stated as

$$\frac{1}{2}\left\{\left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial z}\right)^2\right\} + g\eta = R \qquad \text{at } z = \eta \tag{II-1-102}$$

in which R is the Bernoulli constant.

(i) The boundary-value problem for wave motion as formulated above is complete. The time-dependency may be removed from the problem formulation by simply adapting a coordinate system that moves with the same velocity as the wave phase speed (Fenton 1988; Fenton and McKee 1990; Sobey et al. 1987). This is equivalent to introducing an underlying current relative to which the wave motion is measured. The current (also called *Stokes' drift velocity* or *Eulerian current*) causes a Doppler shift of the apparent wave period measured relative to a stationary observer or gauge. The underlying current velocity must therefore also be known in order to solve the wave problem in the steady (moving) reference frame.

(j) Fenton's solution method uses the Fourier cosine series in kx to the governing equations. It is clearly an approximation, but very accurate, since results of this theory appear not to be restricted to any water depths. $\epsilon = kH/2$ is the expansion parameter replacing ka in the Stokes wave theory. The dependent variable is the stream function Ψ represented by a Fourier cosine series in kx, expressed up to the Nth order as

$$\Psi(x,z) = -\bar{u}(z+d) + \left(\frac{g}{k^3}\right)^{\frac{1}{2}} \sum_{j=1}^{N} B_j \frac{\sinh jk(z+d)}{\cosh jkd} \cos jkx \tag{II-1-103}$$

where the B_j are dimensionless Fourier coefficients. The truncation limit of the series N determines the order of the theory. The nonlinear free-surface boundary conditions are satisfied at each of $M+1$ equi-spaced points on the surface. Wave height, wave period, water depth, and either the mean Eulerian velocity or the Stokes drift velocity must be specified to obtain a solution.

(k) The solution is obtained by numerically computing the N Fourier coefficients that satisfy a system of simultaneous equations. The numerical solution solves a set of $2M+6$ algebraic equations to find unknown Fourier coefficients. The problem is uniquely specified when $M = N$ and overspecified when $M > N$. Earlier wave theories based on stream function consider the overspecified case and used a least-squares method to find the coefficients. Fenton was the first to consider the uniquely specified case and used the collocation method to produce the most accurate and computationally efficient solution valid for any water depth.

(l) An initial estimate is required to determine the $M+N+6$ variables. The linear theory provides this initial estimate for deep water. In relatively shallow water, additional Fourier components are introduced. An alternative method is used in the shallow-water case by increasing the wave height in a number of steps. Smaller heights are used as starting solutions for subsequent higher wave heights. This approach eliminates the triple-crested waves reported by others (Huang and Hudspeth 1984; Dalrymple and Solana 1986).

(m) Sobey et al. (1987) compared several numerical methods for steady water wave problems, including Fenton's. Their comparison indicated that accurate results may be obtained with Fourier series of 10 to 20 terms, even for waves close to breaking. Comparisons with other numerical methods and experimental data (Fenton and McKee 1990; Sobey 1990) showed that results from Fenton's theory and experiments agree consistently and better than results from other theories for a wide range of wave height, wave period, and water depth. Based on these comparisons, Fenton and McKee (1990) define the regions of validity of Stokes and cnoidal wave theory as

$$\frac{L}{d} = 21.5\ e^{\left(-1.87\frac{H}{d}\right)} \tag{II-1-104}$$

(n) The cnoidal theory should be used for wavelengths longer than those defined in this equation. For shorter waves, Stokes' theory is applicable. Fenton's theory can be used over the entire range, including obtaining realistic solutions for waves near breaking.

(o) In water of finite depth, the greatest (unbroken) wave that could prevail as a function of both wavelength and depth is determined by Fenton and McKee (1990) as

$$\frac{H}{d} = \frac{0.141063\frac{L}{d} + 0.0095721\left(\frac{L}{d}\right)^2 + 0.0077829\left(\frac{L}{d}\right)^3}{1.0 + 0.078834\frac{L}{d} + 0.0317567\left(\frac{L}{d}\right)^2 + 0.0093407\left(\frac{L}{d}\right)^3}\text{a} \tag{II-1-105}$$

(p) The leading term in the numerator of this equation is the familiar steepness limit for short waves in deep water. For large values of L/d (i.e., shallow-water waves), the ratio of cubic terms in the above equation approaches the familiar 0.8 value, a limit for depth-induced breaking of the solitary waves. Therefore, the above equation may also be used as a guide to delineate unrealistic waves in a given water depth.

(q) The formulas for wave kinematics, dynamics, and wave integral properties for Fenton's theory have been derived and summarized (Sobey et al.1987; Klopman 1990). Only the engineering quantities of interest including water particle velocities, accelerations, pressure, and water surface elevation defined relative to a Eulerian reference frame are provided here.

(r) The horizontal and vertical components of the fluid particle velocity are

$$u(x,z) = \frac{\partial \Psi}{\partial z} = -\bar{u} + \left(\frac{g}{k}\right)^{\frac{1}{2}} \sum_{j=1}^{N} jB_j\ \frac{\cosh jk(z+d)}{\cosh jkd}\ \cos jkx \tag{II-1-106}$$

$$w(x,z) = -\frac{\partial \Psi}{\partial x} = \left(\frac{g}{k}\right)^{\frac{1}{2}} \sum_{j=1}^{N} jB_j\ \frac{\sinh jk(z+d)}{\cosh jkd}\ \sin jkx \tag{II-1-107}$$

(s) Fluid particle accelerations in the horizontal and vertical directions are found by differentiating the velocities and using the continuity equation. These component accelerations are

$$\begin{aligned} a_x\ (x,z) &= \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \\ a_z(x,z) &= \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = u\frac{\partial u}{\partial z} - w\frac{\partial u}{\partial x} \end{aligned} \tag{II-1-108}$$

where

$$\frac{\partial u}{\partial x} = -\left(\frac{g}{k}\right)^{\frac{1}{2}} \sum_{j=1}^{N} j^2 B_j \, \frac{\cosh jk(z+d)}{\cosh jkd} \, \sin jkx \tag{II-1-109}$$

$$\frac{\partial u}{\partial z} = \left(\frac{g}{k}\right)^{\frac{1}{2}} \sum_{j=1}^{N} j^2 B_j \, \frac{\sinh jk(z+d)}{\cosh jkd} \, \cos jkx \tag{II-1-110}$$

(t) The instantaneous water surface elevation η(x) and water particle pressure are given by

$$\eta(x) = \frac{1}{2} a_N \cos Nkx + \sum_{j=1}^{N-1} a_j \cos jkx$$

$$p(x,z) = \rho(R - gd - gz) - \frac{1}{2}\rho(u^2 + w^2) \tag{II-1-111}$$

(u) Integral properties of periodic gravity waves, including wave potential and kinetic energy, wave momentum and impulse, wave energy flux and wave power, and wave radiation stresses obtained by Klopman (1990) and Sobey et al. (1987) are listed in the Leenknecht, Szuwalski, and Sherlock (1992) documentation.

(v) A computer program developed by Fenton (1988) has recently been implemented in the ACES package. The ACES implementation facilitates use of Fenton's theory to applications in deep water and finite-depth water. It uses Fourier series of up to 25 terms to describe a wave train and provides information about various wave quantities. The output includes wave estimates for common engineering parameters including water surface elevation, wave particle kinematics, and wave integral properties as functions of wave height, period, water depth, and position in the wave form.

(w) The wave is assumed to co-exist on a uniform co-flowing current, taken either as the mean Eulerian current or mean mass transport velocity. At a given point in the water column, wave kinematics are tabulated over two wavelengths, and vertical distribution of the selected kinematics under the wave crest are graphically displayed. ACES implementation of Fenton's theory and its input/output requirements, computations, and examples are described in detail in the ACES documentation manual (Leenknecht, Szuwalski, and Sherlock 1992).

(x) Figure II-1-18 illustrates the application of Fenton's theory. This case represents shallow-water (10-m) conditions and wave height and period of 5 m and 10 sec, respectively. Surface elevation, horizontal velocity, and pressure over two wavelengths is shown graphically in Figure II-1-18. The ACES documentation includes guidance on proper use of Fenton's theory.

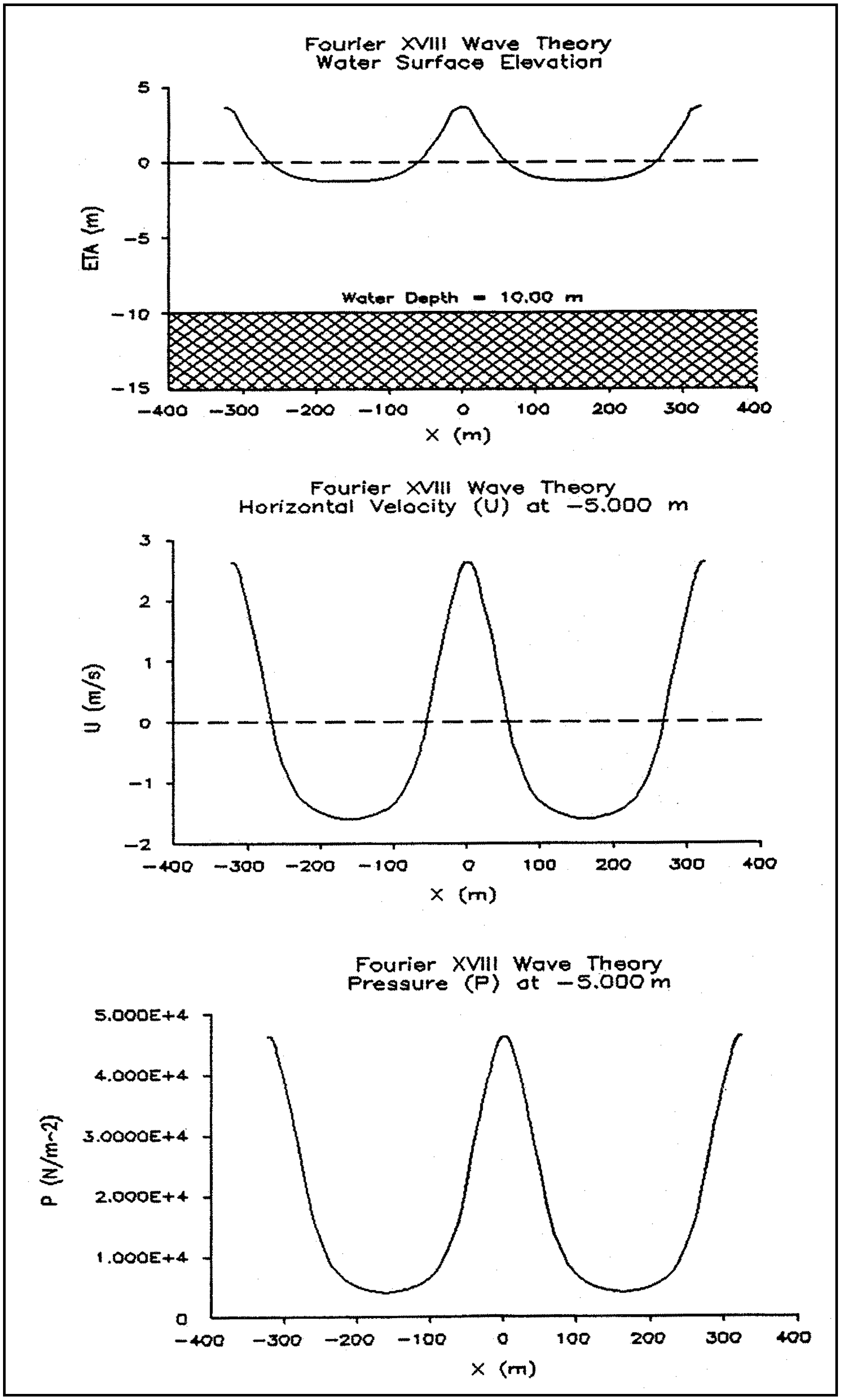

Figure II-1-18. Surface elevation, horizontal velocity, and pressure in 10-m depth (using Fenton's theory in ACES)

f. Wave breaking.

(1) Wave height is limited by both depth and wavelength. For a given water depth and wave period, there is a maximum height limit above which the wave becomes unstable and breaks. This upper limit of wave height, called *breaking wave height*, is in deep water a function of the wavelength. In shallow and transitional water it is a function of both depth and wavelength. Wave breaking is a complex phenomenon and it is one of the areas in wave mechanics that has been investigated extensively both experimentally and numerically.

(2) Researchers have made some progress over the last three decades in the numerical modeling of waves close to breaking (Longuet-Higgins and Fenton 1974; Longuet-Higgins 1974; 1976; Schwartz 1974; Dalrymple and Dean 1975; Byatt-Smith and Longuet-Higgins 1976; Peregrine 1976; Cokelet 1977; Longuet-Higgins and Fox 1977; Longuet-Higgins 1985; Williams 1981; 1985). These studies suggest the limiting wave steepness to be $H/L = 0.141$ in deep water and $H/d = 0.83$ for solitary waves in shallow water with a corresponding solitary wave celerity of $c/(gd)^{1/2} = 1.29$.

(3) Dalrymple and Dean (1975) investigated the maximum wave height in the presence of a steady uniform current using the stream function theory. Figure II-1-19 shows the influence of a uniform current on the maximum wave height where T_c is the wave period in a fixed reference frame and U is the current speed.

(4) The treatment of wave breaking in the propagation of waves is discussed in Part II-3. Information about wave breaking in deep and shoaling water and its relation to nearshore processes is provided in Part II-4.

g. Validity of wave theories.

(1) To ensure their proper use, the range of validity for various wave theories described in this chapter must be established. Very high-order Stokes theories provide a reference against which the accuracy of various theories may be tested. Nonlinear wave theories better describe mass transport, wave breaking, shoaling, reflection, transmission, and other nonlinear characteristics. Therefore, the usage of the linear theory has to be carefully evaluated for final design estimates in coastal practice. It is often imperative in coastal projects to use nonlinear wave theories.

(2) Wave amplitude and period may sometimes be estimated from empirical data. When data are lacking or inadequate, uncertainty in wave height and period estimates can give rise to a greater uncertainty in the ultimate answer than does neglecting the effect of nonlinear processes. The additional effort necessary for using nonlinear theories may not be justified when large uncertainties exist in the wave data used for design. Otherwise, nonlinear wave theories usually provide safer and more accurate estimates.

(3) Dean (1968, 1974) presented an analysis by defining the regions of validity of wave theories in terms of parameters H/T^2 and d/T^2 since T^2 is proportional to the wavelength. Le Méhauté (1976) presented a slightly different analysis (Figure II-1-20) to illustrate the approximate limits of validity for several wave theories, including the third- and fourth-order theories of Stokes. In Figure II-1-20, the fourth-order Stokes theory may be replaced with more popular fifth-order theory, since the latter is often used in applications. Both Le Méhauté and Dean recommend cnoidal theory for shallow-water waves of low steepness, and Stokes' higher order theories for steep waves in deep water. Linear theory is recommended for small steepness H/T^2 and small U_R values. For low steepness waves in transitional and deep water, linear theory is adequate but other wave theories may also be used in this region. Fenton's theory is appropriate for most of the wave parameter domain. For given values of H, d, and T, Figure II-1-20 should be used as a guide to select an appropriate wave theory.

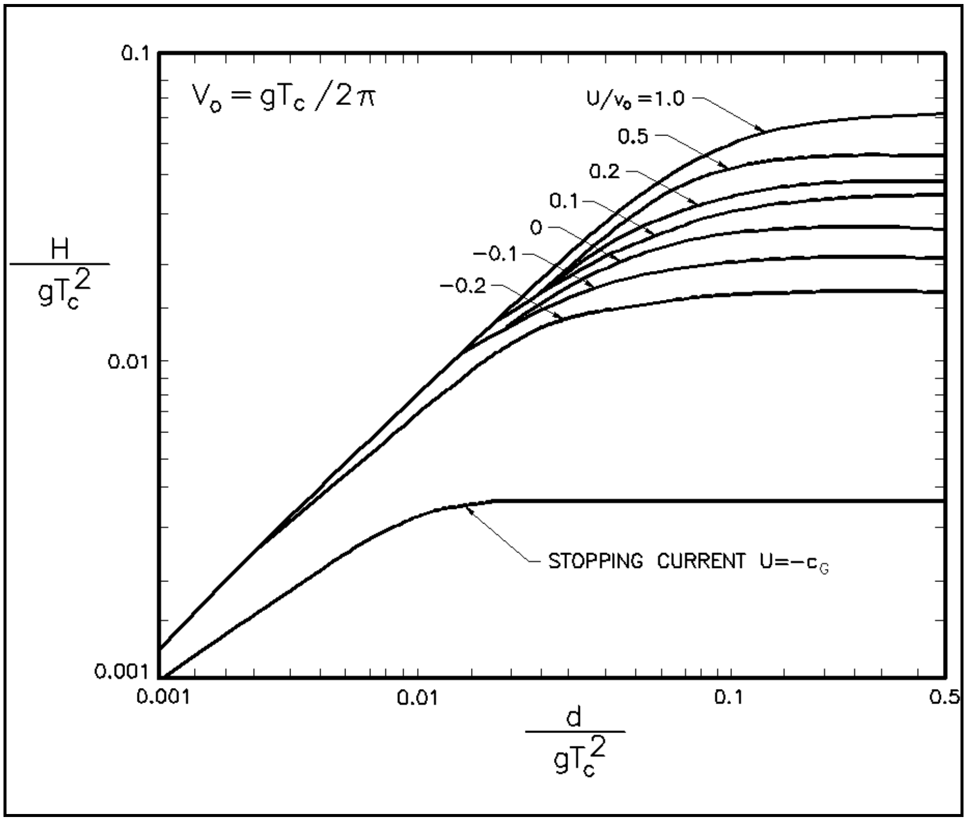

Figure II-1-19. Influence of a uniform current on the maximum wave height (Dalrymple and Dean 1975)

(4) It is necessary to know the limiting value of wave heights and wave steepness at different water depths to establish range of validity of any wave theory that uses a Stokes-type expansion. This is customarily done by comparing the magnitude of each successive term in the expansion. Each should be smaller than the term preceding it. For example, if the second term is to be less than 1 percent of the first term in the Stokes second-order theory, the limiting wave steepness is

$$\frac{H}{L} \le \frac{1}{80} \frac{\sinh^3 kd}{\cosh kd \, (3 + 2 \sinh^2 kd)} \qquad \text{(II-1-112)}$$

(5) If the third-order term is to be less than 1 percent of the second-order term, the limiting wave steepness is

$$\frac{H}{L} \le \frac{1}{7} \frac{\sinh^3 kd}{\sqrt{1 + 8 \cosh^3 kd}} \qquad \text{(II-1-113)}$$

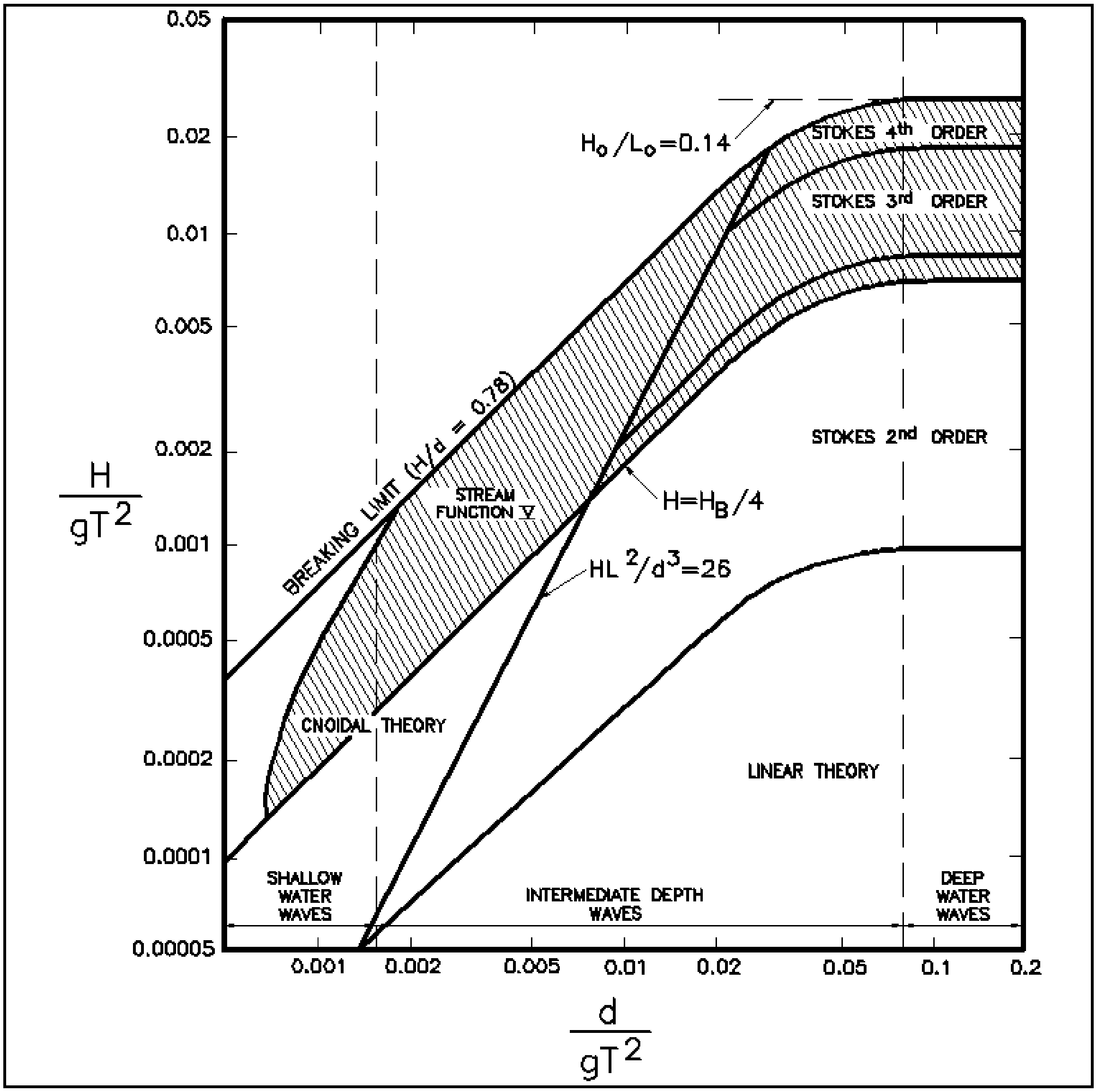

Figure II-1-20. Ranges of suitability of various wave theories (Le Méhauté 1976)

(6) Similarly, using the fifth-order expansion, the asymptotes to Stokes third-order theory are $H/L_0 < 0.1$ and $H/d < 3/4(kd)^2$ for deep water and shallow water, respectively. This allows the range of Stokes' theory to be expanded by adding successively smaller areas to the domain of linear theory in Figure II-1-20 until the breaking limit is reached. The fifth-order Stokes theory gets close enough to the breaking limit, and higher order solutions may not be warranted. Laitone (1962) suggests a shallow-water limit on Stokes' theory by setting the Ursell number U_R equal to 20. For an Ursell number of approximately 20, Stokes' theory approaches the cnoidal theory.

(7) The magnitude of the Ursell number U_R (sometimes also called the *Stokes number*) shown in Figure II-1-20 may be used to establish the boundaries of regions where a particular wave theory should be used. Stokes (1847) noted that this parameter should be small for long waves. An alternative, named the *Universal parameter* (U_p), has recently been suggested (Goda 1983) for classification of wave theories.

(8) Limits of validity of the nonlinear (higher-order) wave theories established by Cokelet (1977) and Williams (1981), are shown in Figure II-1-21. Regions where Stokes waves (short waves) and cnoidal and

solitary waves (long waves) are valid are also shown in this figure. The breaking limit for solitary waves $H_b^W = 0.833$ established by Williams (1981) and the limiting height designated as H_b^F determined by Cokelet (1977) are also shown on Figure II-1-21. The line between short and long waves corresponds to a value of the Ursell number $U_R \approx 79$. This theoretical partition agrees with data from Van Dorn (1966).

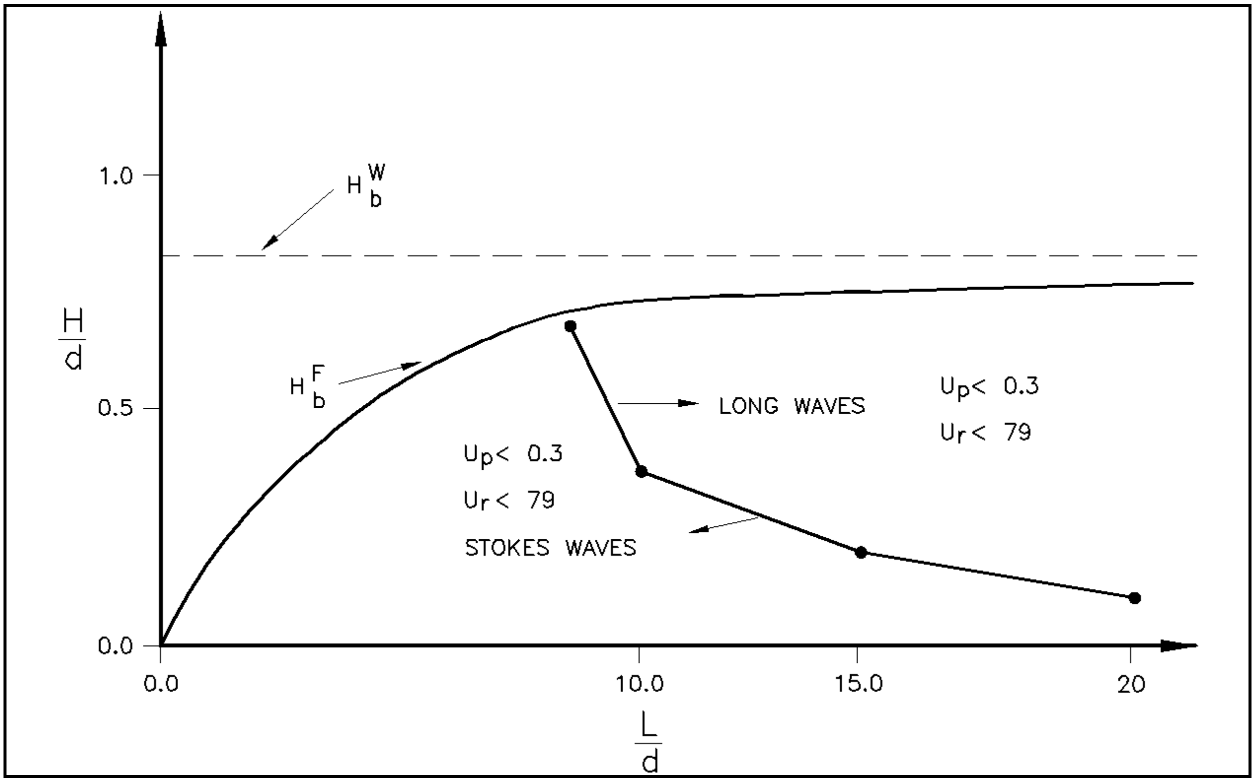

Figure II-1-21. Grouping of wind waves based on universal parameter and limiting height for steep waves

II-1-3. Irregular Waves

a. Introduction.

(1) In the first part of this chapter, waves on the sea surface were assumed to be nearly sinusoidal with constant height, period and direction (i.e., monochromatic waves). Visual observation of the sea surface (as in the radar image of the entrance to San Francisco Bay in Figure II-1-22) and measurements (such as in Figure II-1-23) indicate that the sea surface is composed of waves of varying heights and periods moving in differing directions. In the first part of this chapter, wave height, period, and direction could be treated as deterministic quantities. Once we recognize the fundamental variability of the sea surface, it becomes necessary to treat the characteristics of the sea surface in statistical terms. This complicates the analysis but more realistically describes the sea surface. The term *irregular waves* will be used to denote natural sea states in which the wave characteristics are expected to have a statistical variability in contrast to *monochromatic waves*, where the properties may be assumed constant. Monochromatic waves may be generated in the laboratory but are rare in nature. "Swell" describes the natural waves that appear most like monochromatic waves in deep water, but swell, too, is fundamentally irregular in nature. We note that the sea state in nature during a storm is always short-crested and irregular. Waves that have travelled far from

EXAMPLE PROBLEM II-1-6

FIND:

Applicable wave theory for waves in (a) and (b). Which of these waves is a long wave?

GIVEN:

(a). d = 15 m, H = 12.2 m, T = 12 sec; (b). d = 150 m, H = 30 m, T = 16 sec.

SOLUTION:

(a) Calculate dimensionless parameters necessary for using Figure II-1-20. These are

$$\frac{d}{gT^2} \approx 0.01$$

$$\frac{H}{gT^2} \approx 0.009$$

$$\frac{H}{d} = 0.8$$

$$\sqrt{gd} \approx 12 \ \frac{m}{\text{sec}}$$

$$U_R \approx 55$$

From Figure II-1-20, the applicable theory is cnoidal.

(b) In a similar fashion, compute

$$\frac{d}{gT^2} \approx 0.06$$

$$\frac{H}{gT^2} \approx 0.01$$

$$\frac{H}{d} = 0.2$$

$$\sqrt{gd} \approx 40 \ \frac{m}{\text{sec}}$$

$$U_R \approx 1.5$$

With these values, Figure II-1-20 indicates the applicable theory is Stokes third- or fifth-order. It is noted that the linear theory is also applicable.

Based on the values of Ursell parameter, neither wave (a) or (b) is a true long wave. Wave (a) may be considered a long wave in comparison to wave (b).